4 LAURENT'S THEOREM FOR COMPLEX FUNCTIONS

Consider the function $f : \mathbb{C} \setminus \{2\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z-2}$. By Taylor's theorem, f has a Taylor series centered at $z_0 = 0$ with neighborhood of convergence $N_2(0)$. That is,

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j \quad \text{on} \quad N_2(0).$$

But this function f is also defined for $|z| > 2$, so it is natural to ask if f can be represented by some (other) series expansion centered at $z_0 = 0$ and convergent for $|z| > 2$. To explore this, note that

$$f(z) = \frac{1}{z-2} = \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right) = \frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{2}{z}\right)^j = \sum_{j=0}^{\infty}\frac{2^j}{z^{j+1}},$$

and this series converges for $\left|\frac{2}{z}\right| < 1$, i.e., for $|z| > 2$. This type of power series, which involves powers of $\frac{1}{z-z_0}$, or equivalently, negative powers of $(z - z_0)$, is known as a *Laurent series*. More generally, the form of a Laurent series centered at a point z_0 involves both positive and negative powers of $(z - z_0)$, that is,

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}, \tag{9.21}$$

and so it consists of *two* infinite series. The first series in (9.21) is often referred to as *the analytic part* and the second series in (9.21) is often referred to as *the singular part*, or *the principal part* of the Laurent series. For the Laurent series expansion to exist at a particular $z \in \mathbb{C}$, both the analytic part and the singular part must be convergent at z. In fact, the analytic part consisting of positive powers of $(z - z_0)$ will converge for all z inside some circle of radius R, while the singular part consisting of negative powers of $(z - z_0)$ will converge for all z *outside* some circle of radius r, as in our example. It is the overlap of the two regions of convergence associated with the analytic part and the singular part that comprises the region of convergence of the Laurent series. This region of overlap is typically the annulus centered at z_0 and denoted by $A_r^R(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$. See Figure 9.3 for an illustration. The situation $R = \infty$ and $r = 0$ corresponds to the case where the Laurent series converges everywhere in $\mathbb{C}$ except possibly at z_0. (In our particular example, we have convergence in the annulus $A_2^\infty(0)$.) Note that since the annulus so described *excludes* the point z_0, it is not necessary that f be differentiable at z_0 for it to have a Laurent series expansion centered there. (In fact, the function f need not be differentiable anywhere within $N_r(z_0)$.)

If $0 < r < r_1 < R_1 < R$ we will refer to the annulus $A_{r_1}^{R_1}(z_0) \subset A_r^R(z_0)$ as a *proper subannulus* of $A_r^R(z_0)$. In a manner similar to that of a convergent

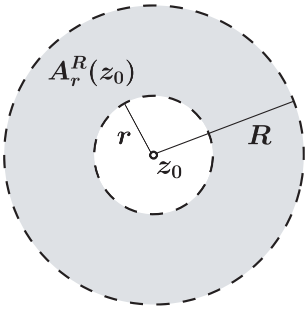

Figure 9.3 *The region of convergence of a Laurent series.*

Taylor series on subsets of its neighborhood of convergence, when a function f has a convergent Laurent series expansion on an annulus $A_r^R(z_0)$, it will converge absolutely on $A_r^R(z_0)$ and uniformly on $\overline{A_{r_1}^{R_1}(z_0)}$ where $A_{r_1}^{R_1}(z_0)$ is any proper subannulus of $A_r^R(z_0)$. To establish this, we must consider the convergence of a given Laurent series a bit more carefully. Since the analytic part of the Laurent series is a power series, it will converge absolutely on its neighborhood of convergence $N_R(z_0)$. Recall that the convergence is uniform on $\overline{N_{R_1}(z_0)}$ for any $0 < R_1 < R$. To analyze the singular part, we define

$$r \equiv \inf\left\{|z - z_0| : \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \ \text{ converges}\right\}.$$

If r fails to exist then the singular part never converges. If r exists then we will show that the singular part converges absolutely for $(\overline{N_r(z_0)})^C$, and uniformly on $(N_{r_1}(z_0))^C$ for any $r_1 > r$. To see this, choose $r_1 > r$. Then as indicated in Figure 9.4 there exists z_1 such that $r \leq |z_1 - z_0| < r_1$ and $\sum_{j=1}^{\infty} \frac{b_j}{(z_1 - z_0)^j}$ converges. Therefore, there exists $M \geq 0$ such that

$$\left|\frac{b_j}{(z_1 - z_0)^j}\right| \leq M \quad \text{for all } \ j \geq 1.$$

Now, for $z \in (N_{r_1}(z_0))^C$ we have

$$\left|\frac{b_j}{(z - z_0)^j}\right| = \left|\frac{b_j}{(z_1 - z_0)^j}\right| \left(\frac{|z_1 - z_0|}{|z - z_0|}\right)^j \leq M \left(\frac{|z_1 - z_0|}{r_1}\right)^j.$$

Since $|z_1 - z_0| < r_1$ this establishes via the Weierstrass M-test that $\sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}$ converges absolutely on $(\overline{N_r(z_0)})^C$ and uniformly on $(N_{r_1}(z_0))^C$. Since $r_1 > r$ was arbitrary, this result holds for any $r_1 > r$. We leave it to the reader to show that $\sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}$ diverges on $N_r(z_0)$. As with a Taylor series, the points on the boundary $C_r(z_0)$ must be studied individually.

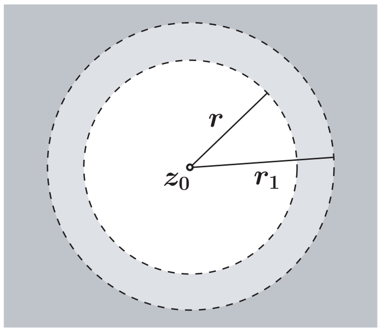

Figure 9.4 *The regions of convergence and divergence of the singular part of a Laurent series.*

► **9.42** *As claimed above, show that* $\sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j}$ *diverges on* $N_r(z_0)$.

The above discussion and exercise establish the following result.

Proposition 4.1 *Suppose* $f : D \to \mathbb{C}$ *has a Laurent series expansion* $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j}$ *on the annulus* $A_r^R(z_0) \subset D$ *(where* $r \geq 0$ *and* R *may be* ∞*). Then for any proper subannulus* $A_{r_1}^{R_1}(z_0) \subset A_r^R(z_0)$ *the given Laurent series expansion for* f *converges absolutely on* $A_{r_1}^{R_1}(z_0)$ *and uniformly on* $\overline{A_{r_1}^{R_1}(z_0)}$.

This result, in turn, implies the following. It will be instrumental in proving part of the key result of this section.

Proposition 4.2 *Let* $\sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j}$ *be the singular part of a Laurent series expansion for* $f : D \to \mathbb{C}$ *on* $A_r^R(z_0) \subset D$. *Then* $\sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j}$ *represents a continuous function on* $(\overline{N_r(z_0)})^C$, *and for any contour* $C \subset (\overline{N_r(z_0)})^C$ *we have*

$$\int_C \left(\sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j} \right) dz = \sum_{j=1}^{\infty} \int_C \frac{b_j}{(z-z_0)^j}\, dz.$$

► **9.43** *Prove the above proposition.*

It is also true, and of great practical value as we will see, that when a Laurent series expansion exists for a function it is unique. To see this, suppose f is differentiable on the annulus $A_r^R(z_0)$ and suppose too that it has a convergent Laurent series there given by

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} \frac{b_k}{(z-z_0)^k}. \tag{9.22}$$

Recall that we established the uniqueness of the Taylor series representation of a function centered at a point by differentiating the series term-by-term; here, we will establish the uniqueness of the Laurent series representation by integrating term-by-term. Let C be any simple closed contour in $A_r^R(z_0)$ with $n_C(z_0) = 1$, and note that $C \subset A_{r_1}^{R_1}(z_0) \subset A_r^R(z_0)$ for some proper subannulus $A_{r_1}^{R_1}(z_0) \subset A_r^R(z_0)$. Also note that for any $k \geq 0$,

$$\begin{aligned}
\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta &= \oint_C \sum_{j=0}^{\infty} a_j (\zeta - z_0)^{j-k-1}\, d\zeta + \oint_C \sum_{j=1}^{\infty} \frac{b_j}{(\zeta - z_0)^{j+k+1}}\, d\zeta \\
&= \sum_{j=0}^{\infty} \oint_C a_j (\zeta - z_0)^{j-k-1}\, d\zeta + \sum_{j=1}^{\infty} \oint_C \frac{b_j}{(\zeta - z_0)^{j+k+1}}\, d\zeta \qquad (9.23) \\
&= 2\pi i\, a_k\, n_C(z_0) \\
&= 2\pi i\, a_k.
\end{aligned}$$

Note in (9.23) above that the integral of each summand corresponding to a_j for $j \neq k$, and for b_j for $j \geq 1$, vanishes due to the integrand having an antiderivative within $N_{R_1}(z_0)$. Also, in integrating the singular part term-by-term, we have applied Proposition 4.2. This shows that each a_k in (9.22) is uniquely determined by f and z_0. A similar argument can be made for the uniqueness of each b_k for $k \geq 1$ in (9.22) by considering the integral $\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{-k+1}}\, d\zeta$ for any fixed $k \geq 1$. We leave this to the reader.

► **9.44** *Establish the uniqueness of the b_k terms in the Laurent series representation (9.22) by carrying out the integral $\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{-k+1}}\, d\zeta$ for any fixed $k \geq 1$.*

With these important facts about an existing Laurent series established, we now state and prove Laurent's theorem, the key result of this section. It is a kind of sibling to Taylor's theorem in complex function theory. It gives conditions under which a function $f : D \to \mathbb{C}$ is guaranteed a Laurent series representation convergent on an annulus $A_r^R(z_0) \subset D$.

Theorem 4.3 (Laurent's Theorem)
Let $f : D \to \mathbb{C}$ be differentiable on the annulus $A_r^R(z_0) = \{z : r < |z - z_0| < R\} \subset D$ (where $r \geq 0$ and R may be ∞). Then $f(z)$ can be expressed uniquely by

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j},$$

on $A_r^R(z_0)$. For any choice of simple closed contour $C \subset A_r^R(z_0)$ with $n_C(z_0) = 1$, the coefficients a_j and b_j are given by

$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta \quad \textit{for} \quad j \geq 0,$$

and

$$b_j = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{-j+1}}\,d\zeta \quad \textit{for} \quad j \geq 1.$$

PROOF As in the proof of Taylor's theorem, we may assume that $z_0 = 0$. Fix $z \in A_r^R(0)$. Choose circles C_1 and C_2 in $A_r^R(0)$ such that both are centered at 0, with radii R_1 and R_2, respectively, satisfying $r < R_2 < |z| < R_1 < R$.

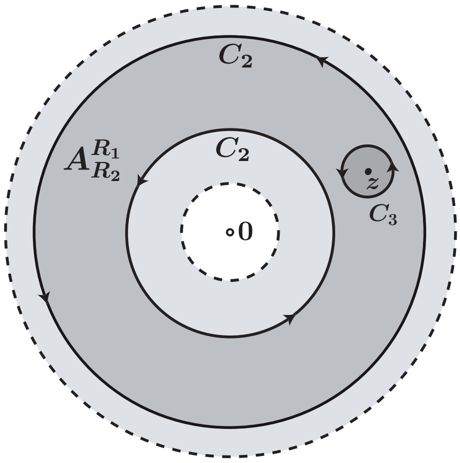

Figure 9.5 *The situation in the proof of Laurent's theorem.*

As indicated in Figure 9.5, choose a third circle C_3, centered at z and having radius R_3, with R_3 small enough that C_3 is contained in $A_r^R(0)$ and does not intersect C_1 or C_2. Then, since $n_{C_1} = n_{C_3} + n_{C_2}$ on $\left(A_r^R(0)\right)^C \cup \{z\}$ (Why?), we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\oint_{C_3} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta && \text{by Theorem 3.6, Chap. 8}\\ &= f(z) + \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta && \text{by Cauchy's formula.}\end{aligned}$$

Solving for $f(z)$ gives

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta\\ &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{z - \zeta}\,d\zeta. \qquad (9.24)\end{aligned}$$

We will show that the first integral on the right-hand side of (9.24) leads to the *analytic part* of the Laurent series expansion for $f(z)$, while the second integral on the right-hand side leads to the *singular part*. The analysis of the first integral proceeds just as in the proof of Taylor's theorem, that is,

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\sum_{j=0}^{N}z^j\,\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta^{j+1}}\,d\zeta+\frac{1}{2\pi i}\oint_{C_1}\left(\frac{z}{\zeta}\right)^{N+1}\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

In this case, however, we can no longer expect that $\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta^{j+1}}\,d\zeta=\frac{f^{(j)}(0)}{j!}$, because f is not necessarily differentiable inside C_1. Therefore, we define a_j by $a_j\equiv\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta^{j+1}}\,d\zeta$, for whatever value this integral takes. This yields

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\sum_{j=0}^{N}a_j z^j+\frac{1}{2\pi i}\oint_{C_1}\left(\frac{z}{\zeta}\right)^{N+1}\frac{f(\zeta)}{\zeta-z}\,d\zeta,$$

and letting $N\to\infty$ as in the proof of Taylor's theorem gives the *analytic part*,

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\sum_{j=0}^{\infty}a_j z^j.$$

To obtain the *singular part*, we apply a similar technique. Consider that for $\zeta\in C_2$,

$$\begin{aligned}\frac{f(\zeta)}{z-\zeta}=\frac{f(\zeta)}{z}\left(\frac{1}{1-\frac{\zeta}{z}}\right)&=\frac{f(\zeta)}{z}\left(1+\frac{\zeta}{z}+\cdots+\left(\frac{\zeta}{z}\right)^N+\left(\frac{\zeta}{z}\right)^{N+1}\frac{1}{1-\frac{\zeta}{z}}\right)\\&=f(\zeta)\left(\frac{1}{z}+\frac{\zeta}{z^2}+\cdots+\frac{\zeta^N}{z^{N+1}}+\left(\frac{\zeta}{z}\right)^{N+1}\frac{1}{z-\zeta}\right)\\&=\sum_{j=1}^{N+1}\frac{\zeta^{j-1}}{z^j}f(\zeta)+\left(\frac{\zeta}{z}\right)^{N+1}\frac{f(\zeta)}{z-\zeta},\end{aligned}$$

so that,

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{z-\zeta}\,d\zeta=\sum_{j=1}^{N+1}\frac{1}{z^j}\left(\frac{1}{2\pi i}\oint_{C_2}\zeta^{j-1}f(\zeta)\,d\zeta\right)+\frac{1}{2\pi i}\oint_{C_2}\left(\frac{\zeta}{z}\right)^{N+1}\frac{f(\zeta)}{z-\zeta}\,d\zeta.$$

In this case, define $b_j\equiv\frac{1}{2\pi i}\oint_{C_2}\zeta^{j-1}f(\zeta)\,d\zeta$, and take the limit as $N\to\infty$ to obtain

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{z-\zeta}\,d\zeta=\sum_{j=1}^{\infty}\frac{b_j}{z^j}.$$

Finally, consider $a_j=\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta^{j+1}}d\zeta$. To show that the conclusion holds for *any* simple closed contour $C\subset A_r^R(z_0)$ with $n_C(z_0)=1$, consider any such C. Then $n_C=1$ on $N_r(0)$, and also $n_{C_1}=n_C$ on $\left(A_r^R(0)\right)^C$, obtaining via Theorem 3.6 from Chapter 8,

$$a_j=\frac{1}{2\pi i}\oint_{C_1}\frac{f(\zeta)}{\zeta^{j+1}}d\zeta=\frac{1}{2\pi i}\oint_{C}\frac{f(\zeta)}{\zeta^{j+1}}d\zeta.$$

A similar argument works to show that $b_j = \frac{1}{2\pi i} \oint_C \zeta^{j-1} f(\zeta)\, d\zeta$. Finally, the uniqueness of the Laurent series expansion was established prior to the statement of the theorem. ♦

► **9.45** *Answer the (Why?) question in the above proof. To do so, exploit the fact that the contours are all circles, and consider the cases z, $\overline{\text{Int}(C_r(0))}$, and $\overline{\text{Ext}(C_R(0))}$ separately.*

In a certain sense, the concept of the Laurent series expansion generalizes that of the Taylor series expansion. It allows for a series representation of $f : D \to \mathbb{C}$ in both negative and positive powers of $(z - z_0)$ in a region that excludes points where f is not differentiable. In fact, if the function f is differentiable on all of $N_R(z_0)$ the Laurent series will reduce to the Taylor series of the function centered at z_0.

One practical use of a function's Laurent series representation is the characterization of that function's *singularities*. We consider this topic next.

5 Singularities

5.1 Definitions

In our motivating example from the last section we discussed the function $f : \mathbb{C} \setminus \{2\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z-2}$. The point $z_0 = 2$ is one where the function f fails to be differentiable, and hence no Taylor series can be found for f centered at $z_0 = 2$. Yet, f is differentiable at points z arbitrarily near to z_0. Such a point z_0 is called a *singularity* of the function f.

Definition 5.1 Suppose $f : D \to \mathbb{C}$ is not differentiable at $z_0 \in \mathbb{C}$. If f is differentiable at some point z in every deleted neighborhood $N_r'(z_0)$, then z_0 is called a **singularity** of f.

There are different types of singularities, and we characterize them now. First, we distinguish between *isolated* and *nonisolated* singularities.

Definition 5.2 Suppose $z_0 \in \mathbb{C}$ is a singularity of $f : D \to \mathbb{C}$. If f is differentiable on some deleted neighborhood $N_r'(z_0) \subset D$ centered at z_0, then z_0 is called an **isolated singularity** of f. Otherwise, z_0 is called a **nonisolated singularity** of f.

Example 5.3 *The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z}$ has an isolated singularity at $z_0 = 0$. The function $\text{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$ has nonisolated singularities at every point $z = x \in (-\infty, 0]$.* ◄

► **9.46** *Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \begin{cases} z & \text{if } \operatorname{Im}(z) \ge 0 \\ -\overline{z} & \text{if } \operatorname{Im}(z) < 0 \end{cases}$. Find the singularities of f and characterize them as isolated or nonisolated. What if $-\overline{z}$ is replaced by $\overline{z}$?*

While we cannot find a Taylor series centered at a singularity of f, we can always find a Laurent series centered at an *isolated* singularity, convergent in an annulus that omits that point. The Laurent series can then be used to further characterize the isolated singularity. There are three subtypes to consider. In the following definition, note that $N_r'(z_0)$ is an annulus $A_0^r(z_0)$ centered at z_0.

Definition 5.4 Let z_0 be an isolated singularity of $f : D \to \mathbb{C}$, and suppose f has Laurent series representation $f(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j}$ on $N_r'(z_0) \subset D$.

1. If $b_j = 0$ for all $j \ge 1$, then f is said to have a **removable singularity** at z_0.
2. If there exists $N \in \mathbb{N}$ such that $b_j = 0$ for $j > N$ but $b_N \neq 0$, then f is said to have a **pole of order** N at z_0. In the special case where $N = 1$, the pole is often referred to as a **simple pole.**
3. Otherwise f is said to have an **essential singularity** at z_0.

It is not hard to show that f has an essential singularity at z_0 if and only if $b_j \neq 0$ for infinitely many values of $j \ge 1$ in its Laurent series expansion, that is, if the Laurent series expansion's singular part is an infinite series.

► **9.47** *Prove the above claim.*

According to this definition, if z_0 is an isolated singularity of $f : D \to \mathbb{C}$ one can determine what type of singularity z_0 is by finding the Laurent series representation for f on an annulus $A_r^R(z_0)$ centered at z_0, and scrutinizing its singular part. However, in practice it is often difficult to obtain the Laurent series directly via Laurent's theorem. Finding alternative means for making this determination is therefore worthwhile, and we do so now. The key will be to exploit the *uniqueness* of the Laurent series representation of a given function on a particular annular region. The following example illustrates the idea.

Example 5.5 *Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z^3} e^z$. We leave it to the reader to confirm that $z_0 = 0$ is an isolated singularity. To determine what type of isolated singularity $z_0 = 0$ is we will find the Laurent series representation of f on $N_1'(0)$. To this end, note that f has the form $f(z) = \frac{1}{z^3} f_0(z)$ where the function f_0 is differentiable on $\mathbb{C}$. Therefore, we can expand f_0 in a Taylor series centered at $z_0 = 0$, namely, $f_0(z) = e^z = \sum_{j=0}^{\infty} \frac{1}{j!} z^j$. We* *then can write,*

$$f(z) = \frac{1}{z^3}\sum_{j=0}^{\infty}\frac{1}{j!}z^j = \sum_{j=0}^{\infty}\frac{1}{j!}z^{j-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!} + \frac{1}{4!}z + \frac{1}{5!}z^2 + \cdots$$

which is convergent on $A_0^\infty(0)$. By the uniqueness property of Laurent series representations, this must be the *Laurent series centered at $z_0 = 0$ for f. Since $b_3 = 1$, and $b_n = 0$ for $n \geq 4$, we see that f has a pole of order* 3 *at $z_0 = 0$.* ◄

► **9.48** *Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{\sin z}{z}$. Show that $z_0 = 0$ is a removable singularity.*

Example 5.6 *Consider the function $f : \mathbb{C} \setminus \{-1, 1\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z^2-1}$. We will determine what kind of singularity $z_0 = 1$ is. Note that $f(z) = \left(\frac{1}{z-1}\right)\left(\frac{1}{z+1}\right)$, and*

$$\begin{aligned} \frac{1}{z+1} = \frac{1}{2+(z-1)} &= \frac{1}{2}\left(\frac{1}{1+\frac{z-1}{2}}\right) \\ &= \frac{1}{2}\sum_{j=0}^{\infty}(-1)^j\left(\frac{z-1}{2}\right)^j = \sum_{j=0}^{\infty}\frac{(-1)^j}{2^{j+1}}(z-1)^j, \end{aligned}$$

which converges for $|z-1| < 2$. Overall then, we have

$$f(z) = \left(\frac{1}{z-1}\right)\left(\frac{1}{z+1}\right) = \sum_{j=0}^{\infty}\frac{(-1)^j}{2^{j+1}}(z-1)^{j-1},$$

and this Laurent series converges on the annulus $A_0^2(1)$. From this we see that $z_0 = 1$ is a simple pole. ◄

► **9.49** *For the function in the above example, show that $z = -1$ is also a simple pole.*

► **9.50** *For $N \in \mathbb{N}$, what kind of singularity is $z = 1$ for the function $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ given by $f(z) = \frac{1}{(z-1)^N}$?*

Example 5.7 *Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = e^{1/z}$. By Taylor's theorem, we know that $e^z = \sum_{j=0}^{\infty}\frac{1}{j!}z^j$, which converges for all $z \in \mathbb{C}$. Therefore the Laurent series for f centered at $z_0 = 0$ is given by*

$$f(z) = e^{1/z} = \sum_{j=0}^{\infty}\frac{1}{j!}\frac{1}{z^j},$$

which converges on $A_0^\infty(0)$. Clearly $z_0 = 0$ is not a removable singularity, nor is it a pole of any order. Therefore, it is an essential singularity. ◄

Finally, we define what it means for a function to have a singularity at infinity.

Definition 5.8 Consider $f : D \to \mathbb{C}$. Let $D^* \equiv \left\{z \in D : \frac{1}{z} \in D\right\}$, and define the function $g : D^* \to \mathbb{C}$ by $g(z) = \frac{1}{z}$. Then the function $f \circ g : D^* \to \mathbb{C}$ is given by $(f \circ g)(z) = f\big(g(z)\big) = f\left(\frac{1}{z}\right)$. We say that f has a **singularity at infinity** if the function $f \circ g$ has a singularity at $z_0 = 0$. The singularity at infinity for f is characterized according to the singularity at $z_0 = 0$ for $f \circ g$.

Example 5.9 *Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^3$. Then the function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $g(z) = \frac{1}{z}$ obtains $f \circ g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $(f \circ g)(z) = f\left(\frac{1}{z}\right) = \frac{1}{z^3}$, which has a pole of order 3 at $z_0 = 0$, and therefore f has a pole of order 3 at infinity.* ◀

▶ **9.51** *Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = e^{1/z}$. Show that the associated singularity of $f \circ g$ at $z_0 = 0$ is removable, and hence that f has a removable singularity at infinity.*

▶ **9.52** *Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{\sin z}{z}$. Show that f has an essential singularity at ∞.*

5.2 Properties of Functions Near Singularities

We now prove some results particular to each type of isolated singularity, and thereby reveal something about how a function behaves near each such point.

Properties of Functions Near a Removable Singularity

Suppose z_0 is a removable singularity of the function $f : D \to \mathbb{C}$. Then for some $N_r'(z_0) \subset D$, the function f has Laurent series representation given by

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{on } N_r'(z_0),$$

and we may extend f to z_0 by defining $F : D \cup \{z_0\} \to \mathbb{C}$ according to

$$F(z) = \begin{cases} f(z) & \text{if } z \in D, \\ a_0 & \text{if } z = z_0. \end{cases}$$

Clearly the function F is differentiable at z_0. (Why?) This leads us to the following theorem.

Theorem 5.10 *Let z_0 be an isolated singularity of $f : D \to \mathbb{C}$.*

a) The point z_0 is a removable singularity of f if and only if f can be extended to a function $F : D \cup \{z_0\} \to \mathbb{C}$ that is differentiable at z_0 and such that $F(z) = f(z)$ on D.

b) The point z_0 is a removable singularity of f if and only if there exists $N_r'(z_0) \subset D$ such that f is bounded on $N_r'(z_0)$.

PROOF Half of the proof of part a) precedes the statement of the theorem, and we leave the remaining half of the proof to the reader. To prove part b), suppose f is differentiable and bounded on $N_r'(z_0) \subset D$, and let $g : N_r(z_0) \to \mathbb{C}$ be given by

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{for } z \neq z_0 \\ 0 & \text{for } z = z_0 \end{cases} .$$

Since f is differentiable on $N_r'(z_0)$, the function g must be too. To see that g is also differentiable at z_0, consider

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0,$$

where the last equality holds since f is assumed bounded on $N_r'(z_0)$. Therefore, g is differentiable on $N_r(z_0)$ and hence has Taylor series representation there given by

$$g(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j = \sum_{j=2}^{\infty} a_j (z - z_0)^j .$$

From this, and the fact that $f(z) = \frac{g(z)}{(z-z_0)^2}$ on $N_r'(z_0)$, we obtain

$$f(z) = \sum_{j=2}^{\infty} a_j (z - z_0)^{j-2} = \sum_{j=0}^{\infty} a_{j+2} (z - z_0)^j \quad \text{on} \quad N_r'(z_0).$$

This last expression must be the Laurent series for f on $N_r'(z_0)$, and hence z_0 is a removable singularity of f. Now assume z_0 is a removable singularity of f. Then f has Laurent series representation of the form

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{on} \quad N_R'(z_0),$$

for some $R > 0$. Since this power series is defined and continuous (in fact, it is differentiable) on $N_R(z_0)$, it must be bounded on $\overline{N_r(z_0)}$ for $0 < r < R$. Therefore $f(z)$, which equals this power series on $N_r'(z_0) \subset \overline{N_r(z_0)}$, must be bounded on $N_r'(z_0) \subset \overline{N_r(z_0)}$. ♦

► **9.53** *Complete the proof of part a) of the above theorem.*

► **9.54** *Prove the following corollaries to the above theorem. Let z_0 be an isolated singularity of $f : D \to \mathbb{C}$.*

a) Then z_0 is a removable singularity if and only if $\lim_{z \to z_0} f(z)$ exists.

b) Then z_0 is a removable singularity if and only if $\lim_{z \to z_0} (z - z_0) f(z) = 0$.

Use of the above theorem is illustrated in the following examples.

Example 5.11 *As we have already seen in exercise 9.48, the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z_0 = 0$. We can extend f to $z_0 = 0$ by assigning it the value $a_0 = 1$ from its Laurent series expansion centered at* 0. *The resulting function $F : \mathbb{C} \to \mathbb{C}$ given by $F(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$ is therefore entire.* ◀

Example 5.12 *Consider the function $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ given by $f(z) = \frac{z^2-1}{z-1}$. Since $f(z)$ equals $z + 1$ on $\mathbb{C} \setminus \{1\}$, it is clearly bounded on $N_1'(1)$. From the above theorem, we conclude that $z_0 = 1$ is a removable singularity of f.* ◀

Properties of Functions Near an Essential Singularity

As established by the following theorem, near an essential singularity a function will take values that are arbitrarily close to any fixed value $w_0 \in \mathbb{C}$.

Theorem 5.13 (The Casorati-Weierstrass Theorem)
Let $z_0 \in \mathbb{C}$ be an essential singularity of $f : D \to \mathbb{C}$. Then, for any $w_0 \in \mathbb{C}$ and any $\epsilon, r > 0$, there exists $z \in N_r'(z_0)$ such that $|f(z) - w_0| < \epsilon$.

PROOF [2] We use proof by contradiction. To this end, let $z_0 \in \mathbb{C}$ be an essential singularity of $f : D \to \mathbb{C}$ and assume the negation of the conclusion. Then there exists $w_0 \in \mathbb{C}$ and $\epsilon, r > 0$ such that $|f(z) - w_0| \geq \epsilon$ on $N_r'(z_0)$. Since z_0 is an isolated singularity of f, there exists $\rho > 0$ such that f is differentiable on $N_\rho'(z_0)$. From this we may conclude that $\frac{1}{f(z)-w_0}$ is differentiable and bounded on $N_\rho'(z_0)$. By Theorem 5.10, z_0 is a removable singularity of $\frac{1}{f(z)-w_0}$ and its Laurent series expansion has the form

$$\frac{1}{f(z) - w_0} = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{on} \quad N_\rho'(z_0). \tag{9.25}$$

Let $m \geq 0$ be the smallest integer such that $a_m \neq 0$. Then (9.25) yields for $z \in N_\rho'(z_0)$,

$$\begin{aligned} \frac{1}{f(z) - w_0} &= \sum_{j=m}^{\infty} a_j (z - z_0)^j \\ &= (z - z_0)^m \sum_{j=m}^{\infty} a_j (z - z_0)^{j-m} \\ &\equiv (z - z_0)^m g(z) \quad \text{on} \quad N_\rho'(z_0), \end{aligned}$$

where $g(z)$ is differentiable on $N_\rho(z_0)$ and $g(z_0) \neq 0$. Also note that there exists $N_s(z_0) \subset N_\rho(z_0)$ such that $g(z) \neq 0$ on $N_s(z_0)$. (Why?) From this, we can write

[2] We follow [Con78].

$$f(z) - w_0 = \frac{1}{(z-z_0)^m} \frac{1}{g(z)} \quad \text{on } N_s'(z_0), \tag{9.26}$$

where $\frac{1}{g(z)}$ is differentiable on $N_s(z_0)$. Therefore, $\frac{1}{g(z)}$ has Taylor expansion

$$\frac{1}{g(z)} = \sum_{j=0}^{\infty} c_j (z-z_0)^j \quad \text{on } N_s(z_0), \quad \text{with } c_0 \neq 0, \quad \text{(Why?)}$$

and (9.26) becomes

$$\begin{aligned} f(z) - w_0 &= \frac{1}{(z-z_0)^m} \frac{1}{g(z)} \\ &= \frac{1}{(z-z_0)^m} \sum_{j=0}^{\infty} c_j (z-z_0)^j \\ &= \sum_{j=0}^{\infty} c_j (z-z_0)^{j-m} \\ &= \frac{c_0}{(z-z_0)^m} + \frac{c_1}{(z-z_0)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-z_0)} + \sum_{j=m}^{\infty} c_j (z-z_0)^{j-m}. \end{aligned}$$

Adding w_0 to both sides of the above equality obtains the Laurent series representation for f centered at z_0 from which we can clearly see that z_0 is a pole of order m. This is a contradiction. ♦

► **9.55** *Answer the two (Why?) questions in the above proof.*

The Casorati-Weierstrass theorem is closely related to another result, often referred to as Picard's theorem, which we will not prove here. Picard's theorem states that if f has an essential singularity at z_0, then in any deleted neighborhood of z_0 the function f takes on every complex value with possibly a single exception.

Properties of Functions Near a Pole

We now characterize the behavior of a function near a pole.

Theorem 5.14 *Suppose $f : D \to \mathbb{C}$ has an isolated singularity at z_0. Then z_0 is a pole of order $N \geq 1$ if and only if for some $N_r'(z_0) \subset D$ there exists a function $f_0 : N_r(z_0) \to \mathbb{C}$ differentiable on $N_r(z_0)$ such that $f_0(z_0) \neq 0$ and $f(z) = \frac{f_0(z)}{(z-z_0)^N}$ for all $z \in N_r'(z_0)$.*

PROOF Suppose a function $f_0 : N_r(z_0) \to \mathbb{C}$ exists as described. Since f_0 is differentiable on $N_r(z_0)$, it has a convergent Taylor series

$$f_0(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j \quad \text{on } N_r(z_0).$$

Then
$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^{j-N}$$
is the Laurent series for f valid on $A_0^r(z_0)$. Since $f_0(0) = a_0 \neq 0$, we see that f has a pole of order N at z_0. Now suppose z_0 is a pole of order $N \geq 1$ associated with the function f. Then there exists a deleted neighborhood $N_r'(z_0) \subset D$ on which f is differentiable and on which f has the Laurent series representation given by
$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{N} \frac{b_j}{(z - z_0)^j}, \tag{9.27}$$
with $b_N \neq 0$. Define $f_0 : N_r(z_0) \to \mathbb{C}$ according to $f_0(z) = (z - z_0)^N f(z)$. Then clearly f_0 has Laurent series representation on $N_r'(z_0)$ given by
$$\begin{aligned} f_0(z) &= (z - z_0)^N \sum_{j=0}^{\infty} a_j (z - z_0)^j + (z - z_0)^N \sum_{j=1}^{N} \frac{b_j}{(z - z_0)^j}, \qquad (9.28) \\ &= \sum_{j=0}^{\infty} a_j (z - z_0)^{j+N} + \sum_{j=1}^{N} b_j (z - z_0)^{N-j}, \\ &= \sum_{j=0}^{\infty} a_j (z - z_0)^{j+N} + b_1 (z - z_0)^{N-1} + b_2 (z - z_0)^{N-2} + \cdots + b_N. \end{aligned}$$
From this we see that f_0 is, in fact, differentiable on $N_r(z_0)$, and that $f_0(z_0) = b_N \neq 0$. Comparing equations (9.27) and (9.28) shows that $f(z) = \frac{f_0(z)}{(z-z_0)^N}$ on $N_r'(z_0)$. ♦

Corollary 5.15 *Suppose $f : D \to \mathbb{C}$ has an isolated singularity at z_0. Then z_0 is a pole if and only if $\lim_{z \to z_0} |f(z)| = \infty$.*

PROOF Suppose z_0 is a pole of order $N \geq 1$ of f. Then there exists a neighborhood $N_r(z_0) \subset D$ and a function $f_0 : N_r(z_0) \to \mathbb{C}$ differentiable on $N_r(z_0)$ such that $f_0(z_0) \neq 0$ and $f(z) = \frac{f_0(z)}{(z-z_0)^N}$ on $N_r'(z_0)$. By continuity there exists a neighborhood $N_s(z_0) \subset N_r(z_0)$ such that $|f_0(z)| \geq c > 0$ on $N_s(z_0)$. Therefore, $|f(z)| \geq \frac{c}{|z-z_0|^N}$ on $N_s'(z_0)$, which implies $\lim_{z \to z_0} |f(z)| = \infty$. Conversely, suppose $\lim_{z \to z_0} |f(z)| = \infty$. Since z_0 is not a removable singularity (Why?), it must either be a pole or an essential singularity. But $\lim_{z \to z_0} |f(z)| = \infty$ implies that for any $M \geq 0$ there is a $\delta > 0$ such that $|f(z)| \geq M$ for $|z - z_0| < \delta$. By the Casorati-Weierstrass theorem, z_0 can't be an essential singularity (Why?), and so it must be a pole. ♦

▶ **9.56** *Answer the two (Why?) questions in the above proof.*

Note the difference between the behavior of a function near a pole as opposed to its behavior near an essential singularity. While approaching a pole, the magnitude of the function's value grows without bound, whereas approaching an essential singularity causes the function's value to go virtually "all over the place" infinitely often as z gets nearer to z_0. The behavior near an essential singularity is singular indeed!

The following theorem points to yet another way in which poles of a function are "less strange" than essential singularities. In fact, if a function f has a pole of order N at z_0, then near z_0 the function f behaves like $\frac{1}{(z-z_0)^N}$ for the same reason that near an isolated zero of order N a differentiable function behaves like $(z-z_0)^N$. The theorem suggests even more, namely, that the reciprocal of a function with an isolated zero of order N at z_0 will be a function with a pole of order N at that point. Also, a type of converse holds as well. The reciprocal of a function f with a pole of order N at z_0 is not quite a function with a zero of order N at z_0. But it can ultimately be made into one, as the proof shows. Ultimately, the poles of f can be seen to be zeros of the reciprocal of f in a certain sense.

Theorem 5.16

a) *Suppose $f : D \to \mathbb{C}$ is differentiable on D and has an isolated zero of order N at z_0. Then there exists $N_r'(z_0) \subset D$ such that the function $g : N_r'(z_0) \to \mathbb{C}$ given by $g(z) \equiv \frac{1}{f(z)}$ has a pole of order N at z_0.*

b) *Suppose $f : D \to \mathbb{C}$ has a pole of order N at z_0. Then there exists $N_r'(z_0) \subset D$ such that the function $g : N_r(z_0) \to \mathbb{C}$ given by $g(z) \equiv \begin{cases} \frac{1}{f(z)} & \text{for } z \in N_r'(z_0) \\ 0 & \text{for } z = z_0 \end{cases}$ has a zero of order N at z_0.*

PROOF To prove part a), note that there exists $h : D \to \mathbb{C}$ differentiable on D with $h(z_0) \neq 0$ such that $f(z) = (z - z_0)^N h(z)$. By continuity of h there exists $N_r(z_0) \subset D$ such that $h(z) \neq 0$ on $N_r(z_0)$. Therefore, the function $g : N_r'(z_0) \to \mathbb{C}$ given by $g(z) = \frac{1}{f(z)} = \frac{1/h(z)}{(z-z_0)^N}$ has a pole of order N at z_0. To prove part b), we may assume there exists a function $f_0 : N_s(z_0) \to \mathbb{C}$ where $N_s(z_0) \subset D$, f_0 is differentiable on $N_s(z_0)$, and $f_0(z_0) \neq 0$ such that $f(z) = \frac{f_0(z)}{(z-z_0)^N}$ on $N_s'(z_0)$. Again, by continuity of f_0 there exists $N_r(z_0) \subset N_s(z_0)$ such that $f_0(z) \neq 0$ on $N_r(z_0)$. Therefore, on $N_r'(z_0)$ we have $\frac{1}{f(z)} = (z - z_0)^N \frac{1}{f_0(z)}$, which implies that z_0 can be defined as a zero of order N for $\frac{1}{f}$. In fact, defining $g : N_r(z_0) \to \mathbb{C}$ by $g(z) = \begin{cases} \frac{1}{f(z)} & \text{for } z \in N_r'(z_0) \\ 0 & \text{for } z = z_0 \end{cases}$ obtains

$g(z) = \begin{cases} (z - z_0)^N \frac{1}{f_0(z)} & \text{for } z \in N_r'(z_0) \\ 0 & \text{for } z = z_0 \end{cases}$. This function g has a zero of order N at z_0 as was to be shown. ♦

It should be noted that in the proof of part *b)* above, something a bit subtle occurs. It is not quite true that the reciprocal of the function f having a pole of order N at z_0 is a function with a zero of order N at z_0. This is because the original function f isn't even defined at z_0, and so the reciprocal of f isn't defined there either, initially. However, it turns out that z_0 is a removable singularity of the reciprocal of f. When assigned the value zero at z_0, the reciprocal of f becomes a differentiable function at z_0 with a zero of multiplicity N there. So while the domain of the function f excludes z_0, we can extend the domain of the function $\frac{1}{f}$ so as to include z_0.

6 The Residue Calculus

6.1 Residues and the Residue Theorem

Let z_0 be an isolated singularity of $f : D \to \mathbb{C}$ and suppose f is differentiable on the annulus $A_r^R(z_0) \subset D$. By Laurent's theorem, f has a Laurent series expansion on $A_r^R(z_0)$,

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}.$$

The coefficient b_1 in this expansion is especially significant.

Definition 6.1 Let z_0 be an isolated singularity of the function $f : D \to \mathbb{C}$. The coefficient b_1 of the Laurent series representation of f on $A_r^R(z_0) \subset D$ is called the **residue** of f at z_0 and is denoted by $\mathrm{Res}(f, z_0)$.

In fact, if $C \subset A_r^R(z_0)$ is a simple closed contour with $n_C(z_0) = 1$, Laurent's theorem tells us that

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Rearranging this formula obtains

$$\oint_C f(z)\, dz = 2\pi i\, b_1, \tag{9.29}$$

and so the value of b_1 is instrumental in evaluating complex integrals. If f is differentiable for all $z \in \mathrm{Int}(C)$, and in particular at z_0, Cauchy's integral theorem implies that this integral will be zero and so b_1 should be zero as well. In fact, for such a function the whole singular part of the Laurent expansion will vanish, leaving just the analytic part as the Taylor series expansion for

f. But for a function f that is *not* differentiable at z_0, we will find that for a simple closed contour C with $n_C(z_0) = 1$ the only term in the Laurent series expansion for f that contributes a nonzero value to the integral in (9.29) is the b_1 term, hence the name "residue." For these reasons, it is of particular interest to be able to compute the residue b_1 easily.

Example 6.2 *Recall the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z^3}\, e^z$ discussed in Example 5.5 on page 494. We found the Laurent series of f centered at $z_0 = 0$ and convergent on $A_0^\infty(0)$ to be*

$$f(z) = \frac{1}{z^3}\sum_{j=0}^{\infty}\frac{1}{j!}z^j = \sum_{j=0}^{\infty}\frac{1}{j!}z^{j-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!} + \frac{1}{4!}z + \frac{1}{5!}z^2 + \cdots .$$

From this, we see that $z_0 = 0$ is a pole of order 3, and $\mathrm{Res}(f, 0) = b_1 = \frac{1}{2}$. ◀

Note that in the above example we found the Laurent series expansion of the function $f(z) = \frac{1}{z^3} f_0(z)$ on $A_0^\infty(0)$ by finding the Taylor series for the differentiable function f_0 on a neighborhood of $z_0 = 0$, and then dividing that Taylor series by z^3. The residue b_1 of the resulting Laurent series corresponded to the second coefficient of the Taylor series of f_0, that is, $\mathrm{Res}(f, 0) = \frac{f_0''(0)}{2!} = \frac{1}{2}$. This is so because the coefficient of z^2 in the Taylor series of f_0, when divided by z^3, becomes the coefficient of the $\frac{1}{z}$ term of the final Laurent series. This is true more generally, and is stated in the following important result.

Theorem 6.3 *Suppose $f : D \to \mathbb{C}$ is differentiable on $A_r^R(z_0)$ and let z_0 be a pole of order N of f where $f_0 : N_R(z_0) \to \mathbb{C}$ is the differentiable function on $N_R(z_0)$ such that $f(z) = \frac{f_0(z)}{(z-z_0)^N}$, and $f_0(z_0) \neq 0$. Then*

$$\mathrm{Res}(f, z_0) = \frac{f_0^{(N-1)}(z_0)}{(N-1)!}.$$

Note that if $N = 1$, then $\mathrm{Res}(f, z_0) = f_0(z_0)$. Also recall that if z_0 is a pole of order N for f, Theorem 5.14 on page 499 showed that we may choose f_0 to be given by $f_0(z) = (z - z_0)^N f(z)$ in such a way that f_0 is differentiable at z_0.

▶ **9.57** *Prove Theorem 6.3.*

▶ **9.58** *Find* $\mathrm{Res}(\frac{\sin z}{z}, 0)$.

▶ **9.59** *Find the residues of the function $f : \mathbb{C}\setminus\{\pm 1\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z^2-1}$ at each of its singularities.*

▶ **9.60** *Prove the following: Suppose the functions f and g are differentiable at z_0, $f(z_0) \neq 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$. Then the function given by $\frac{f(z)}{g(z)}$ has a simple pole at z_0, and* $\mathrm{Res}\left(\frac{f(z)}{g(z)}, z_0\right) = \frac{f(z_0)}{g'(z_0)}$.

Example 6.4 *Consider $f : N'_{\frac{3}{2}}(1) \to \mathbb{C}$ given by $f(z) = \frac{1}{z^2-1}$. We will use equality* (9.29) *and Theorem 6.3 to evaluate*

$$\oint_C \frac{dz}{z^2-1}, \quad \textit{where } C = C_1(1).$$

Note that $z_0 = 1$ is a simple pole of f, and that $f_0 : N_{\frac{3}{2}}(1) \to \mathbb{C}$ given by $f_0(z) = \frac{1}{z+1}$ is differentiable on $N_{\frac{3}{2}}(1)$ with $f_0(1) = \frac{1}{2} \neq 0$, and $f(z) = \frac{f_0(z)}{z-1}$. By Theorem 6.3, $\text{Res}(f, 1) = f_0(1) = \frac{1}{2}$, *and therefore equality* (9.29) *yields*

$$\oint_C \frac{dz}{z^2-1} = 2\pi\, i \,\text{Res}(f, 1) = \pi\, i.$$ ◄

► **9.61** *Evaluate* $\oint_C \frac{dz}{z^2-1}$ *where* $C = C_1(-1)$.

► **9.62** *Evaluate* $\oint_C \frac{dz}{z^2-1}$ *where* $C = [2i + 2, 2i - 2, -2i - 2, -2i + 2, 2i + 2]$.

The following important theorem generalizes the idea illustrated in the previous example. It highlights the great practicality in using residues to compute complex integrals.

Theorem 6.5 (The Residue Theorem)
Let $\widetilde{D} \subset \mathbb{C}$ be open and connected and suppose $\{z_1, \ldots, z_M\} \subset \widetilde{D}$. Let $D \equiv \widetilde{D} \setminus \{z_1, \ldots, z_M\}$ and suppose $f : D \to \mathbb{C}$ is differentiable on D. If $C \subset D$ is any closed contour such that $n_C(w) = 0$ for all $w \in \widetilde{D}^C$, then,

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{M} n_C(z_j)\text{Res}(f, z_j).$$

PROOF Since f is differentiable on $D \equiv \widetilde{D} \setminus \{z_1, \ldots, z_M\}$, we may choose $r > 0$ small enough such that the deleted neighborhoods $N'_r(z_j) \subset D \setminus C$ for $j = 1, \ldots, M$ are disjoint (see Figure 9.6). Also, choose circles C_j parametrized by $\zeta_j : [0, 2\pi] \to N'_r(z_j)$ with

$$\zeta_j(t) = \tfrac{1}{2}\, r\, e^{i n_j t}, \quad \text{where } n_j \equiv n_C(z_j).$$

Then it follows that

(1) $n_{C_j}(z_j) = n_j = n_C(z_j)$,

(2) $n_C(w) = 0 = \sum_{j=1}^{M} n_{C_j}(w)$ for $w \in \widetilde{D}^C$,

which by Theorem 3.6 of Chapter 8 implies

$$\oint_C f(z)dz = \sum_{j=1}^{M} \oint_{C_j} f(z)dz.$$

But since

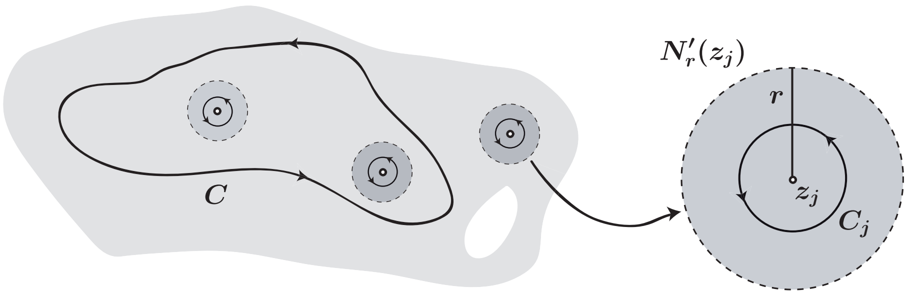

Figure 9.6 *The situation in the proof of the residue theorem.*

$$\begin{aligned}\oint_{C_j} f(z)dz &= n_j \oint_{C_{\frac{r}{2}}(z_j)} f(z)dz \\ &= n_j\, 2\pi i\, \mathrm{Res}(f, z_j) \\ &= 2\pi i\, n_C(z_j)\, \mathrm{Res}(f, z_j),\end{aligned}$$

the theorem follows. ♦

The above theorem can also be thought of as a generalized Cauchy's integral theorem in that, for a function $f : D \to \mathbb{C}$ integrated along contour $C \subset D$, it provides a formula for the value of the integral in the more complicated case where there are finitely many isolated singularities in $D \setminus C$. When no such singularities are present, it reduces to Cauchy's integral theorem.

Example 6.6 *We will evaluate the integral* $\oint\limits_C \frac{dz}{z^2-1}$ *for the contour* C *shown in Figure 9.7.*

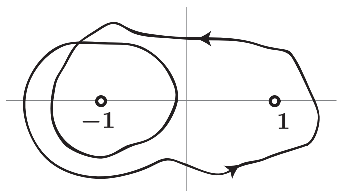

Figure 9.7 *The contour* C *of Example 6.6.*

Note that $f : \mathbb{C} \setminus \{\pm 1\} \to \mathbb{C}$ *given by* $f(z) = \frac{1}{z^2-1}$ *is differentiable on* $D \equiv \mathbb{C} \setminus \{\pm 1\}$. *Since* $z = \pm 1 \notin C$, *and* $n_C(1) = 1$ *and* $n_C(-1) = 2$ *can be easily verified using the visual inspection technique justified in the Appendix of Chapter 8, we have*

$$\begin{aligned}\oint\limits_C \frac{dz}{z^2-1} &= 2\pi\, i\left[n_C(-1)\mathrm{Res}\left(\tfrac{1}{z^2-1}, -1\right) + n_C(1)\mathrm{Res}\left(\tfrac{1}{z^2-1}, 1\right)\right] \\ &= 2\pi\, i\left[2\left(-\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right)\right] = -\pi i.\end{aligned}$$

◄

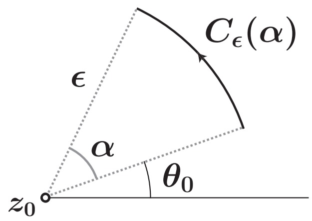

Figure 9.8 *A fractional circular arc.*

The Fractional Residue Theorem

Suppose z_0 is a simple pole of the function $f : D \to \mathbb{C}$. Let θ_0 be some fixed angle, and suppse $\epsilon > 0$. Let $C_\epsilon(\alpha) \equiv \{z_0+\epsilon e^{it} : \theta_0 \le t \le \theta_0+\alpha\}$ be the circular arc subtending α radians of the circle of radius ϵ centered at z_0 as shown in Figure 9.8. We will sometimes need to evaluate integrals of the form

$$\lim_{\epsilon\to 0}\Bigg(\int\limits_{C_\epsilon(\alpha)} f(z)\,dz\Bigg). \tag{9.30}$$

In such cases, the following result will be useful.

Theorem 6.7 *Suppose z_0 is a simple pole of the function $f : D \to \mathbb{C}$, and let $C_\epsilon(\alpha)$ be defined as above. Then,*

$$\lim_{\epsilon\to 0}\Bigg(\int\limits_{C_\epsilon(\alpha)} f(z)\,dz\Bigg) = i\alpha\,\mathrm{Res}(f, z_0).$$

Note that if $\alpha = 2\pi$ this result is consistent with the residue theorem. And yet, the proof of this result requires much less than that of the residue theorem.

PROOF [3] Since z_0 is a simple pole, we know that on some neighborhood $N_r(z_0)$ we have

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots = \frac{b_1}{z-z_0} + g(z),$$

where g is differentiable on $N_r(z_0)$. If $\epsilon > 0$ is small enough, then $C_\epsilon(\alpha) \subset N_r(z_0)$, and

$$\int\limits_{C_\epsilon(\alpha)} f(z)\,dz = b_1 \int\limits_{C_\epsilon(\alpha)} \frac{dz}{z-z_0} + \int\limits_{C_\epsilon(\alpha)} g(z)\,dz. \tag{9.31}$$

Note that

$$\int\limits_{C_\epsilon(\alpha)} \frac{dz}{z-z_0} = \int_{\theta_0}^{\theta_0+\alpha} \frac{i\epsilon e^{it}}{\epsilon e^{it}}\,dt = i\alpha,$$

[3]We follow [MH98].

and so all that remains is to show that the second integral on the right of (9.31) vanishes as $\epsilon \to 0$. To establish this, note that $g(z)$ must be bounded on $C_\epsilon(\alpha)$, and so

$$\left| \int_{C_\epsilon(\alpha)} g(z)\, dz \right| \le M\, L_{C_\epsilon(\alpha)} \quad \text{for some } \; M \ge 0.$$

From this we obtain $\lim\limits_{\epsilon \to 0} \Big(\int_{C_\epsilon(\alpha)} g(z)\, dz \Big) = 0$, and the theorem is proved. ♦

6.2 Applications to Real Improper Integrals

The residue theorem is an especially useful tool in evaluating certain real improper integrals, as the following examples serve to illustrate.

Example 6.8 *We begin by applying the technique to a real improper integral whose value we already know, namely,* $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$. *Traditionally this result is obtained by simply recognizing* $\tan^{-1} x$ *as the antiderivative to* $\frac{1}{x^2+1}$. *Here, we'll begin by defining the contour* $C = \{C_1, C_2\}$, *where* $C_1 = [-R, R]$ *for* $R > 1$ *and* C_2 *is the semicircle going counterclockwise from* $(R, 0)$ *to* $(-R, 0)$ *as illustrated in Figure 9.9.*

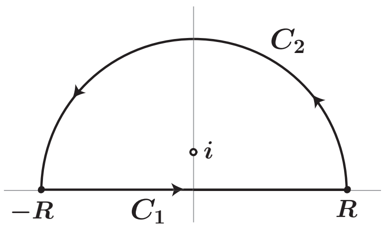

Figure 9.9 *The contour of Example 6.8.*

We will evaluate

$$\oint_C \frac{dz}{z^2+1} = \int_{C_1} \frac{dz}{z^2+1} + \int_{C_2} \frac{dz}{z^2+1}.$$

By the residue theorem, we know

$$\oint_C \frac{dz}{z^2+1} = 2\pi\, i \mathrm{Res}\Big(\frac{1}{z^2+1}, i\Big).$$

To find $\mathrm{Res}\big(\frac{1}{z^2+1}, i\big)$, *we write*

$$\frac{1}{z^2+1} = \frac{\frac{1}{z+i}}{z-i},$$