II.—A GENERAL NOTATION FOR THE LOGIC OF RELATIONS.

BY C. D. BROAD.

§ 1. THE object of the present paper is to offer a consistent system of notation which shall be extensible to relations of any degree of polyadicity. The notation for the logic of relations developed in *Principia Mathematica*, so far as that work has gone, is highly convenient for dyadic relations, which alone have as yet been treated. But it is not readily extensible to triadic and higher relations.

Doubtless these will be dealt with by Dr. Whitehead in the fourth volume, which is to treat of geometry. But the necessity for a satisfactory notation for relational propositions in general is urgent. Work of the utmost importance, such as Mr. Robb's *Theory of Time and Space*, cries aloud for translation into symbolic logic; and I doubt if any great progress in this most promising direction can be made until logic has developed a satisfactory notation for relations of high degrees of polyadicity and for their associated logical functions. For this reason I venture to put forward the following sketch in the hope that it may be at least temporarily useful till Dr. Whitehead publishes the fourth volume of *Principia*.

I am not acquainted with any other attempts in this direction except the notation created *ad hoc* by Whitehead in his *Mathematical Concepts of the Material World* (Proc. Roy. Soc., 1906). This notation, though convenient for its purpose, does not claim to be closely connected with the notation already worked out for dyadic relations.

No special logical or philosophical theory underlies the notation which I offer in the present article, though I believe that the notion of a relational *complex* as distinct from a relational proposition has an important bearing on the theory of judgment.

§ 2. *Complexes and Propositions.*—I begin by distinguishing between relational complexes and relational propositions. Let R be any relation, and, for simplicity, let it be dyadic. Then I denote by the formula R(x, y) what I call a *relational complex.* Suppose that R = the relation of loving, that

Mind n.s. vol. 27, no.107 (Jul 1918), 284-303

x = Smith, and y = Jones; then R (Smith, Jones) stands for what is denoted by the phrase *Smith's love for Jones.* Similarly R (Jones, Smith) stands for what is denoted by *Jones's love for Smith.*

These are clearly not propositions. We *seem* to be able to consider such complexes and to make assertions about them even if we know that Smith does not love Jones, or are doubtful on the point. Take, *e.g.*, the conditional proposition: it would be a good thing if Smith loved Jones. It might be held that this ascribes a predicate to a relational complex without asserting the relational proposition corresponding to the complex.

I propose to symbolise the corresponding relational *proposition* by the formula

R (Smith, Jones)!

The difference between the assertorical proposition: it *is* a good thing that Smith loves Jones, and the conditional proposition: it *would be* a good thing if Smith loved Jones would then seem to be that the first is

R (Smith, Jones)! and R (Smith, Jones) is good

whilst the second is merely

R (Smith, Jones) is good.

Again, it might seem a possible view that ethical predicates always apply to relational *complexes* without regard to the truth or falsity of the corresponding relational *propositions*, and that this is a peculiarity of such predicates. But this, like the question whether relational complexes be in any sense real when the corresponding relational propositions are false, is a philosophical question which need not trouble us for the present purpose. All that we need say for the present is (*a*) that there is a recognisable difference between R(x, y) and R(x, y)!; (*b*) that the question whether ! is wholly logical (*i.e.*, belongs wholly to *objects* of thought), or wholly psychological (*i.e.*, belongs wholly to mental acts), or is something connected with the relation between acts and objects, needs careful consideration; and (*c*) that its connexion with Russell's and Frege's assertion-symbol needs further investigation. It cannot, I think, be identical with the assertion-symbol; for this applies to propositions, whilst ! turns a complex into a proposition.

§ 3. *Complexes and Functions.*—I next wish to point out that R(x, y) is strictly a *function* of x and y in the sense in which *function* is used in mathematics, whilst what Russell calls a propositional function is not in this sense a function at all. *E.g.*, x^2 means the same as *the square of* x, and $x \times y$ means the same as *the product of* x *by* y, just as $R(x, y)$ might stand for what is denoted by *the love of* x *for* y.

But a propositional function for Russell seems to mean a proposition whose terms are variables instead of constants. It seems better to avoid the word *function* altogether in this connexion, since, in the strict sense of the phrase, Smith's love of Jones is as much a function of Smith and Jones as x's love of y is a function of x and y. There are really three distinctions to be considered and symbolised both among complexes and among propositions: (i) the definite complex or proposition (Smith's love of Jones—Smith loves Jones); (ii) a variable instance of the same *form* (x's love of y—x loves y); (iii) the form itself. This, I take it, is what Russell symbolises by $\phi\hat{x}$. By propositional function Russell appears to mean sometimes a form and sometimes a variable instance of a form.

I shall symbolise the form of a relational complex involving R by R(– , – , – ,) when there are as many blanks as the relation has degrees of polyadicity. A variable instance of the form can be symbolised by $R(x, y, z)$. A definite individual instance can be symbolised by R (Smith, Jones, Brown). Corresponding to these complex-symbols there will be the propositional-symbols

R (– , – , –) !
$R\,(x, y, z)$!
R (Smith, Brown, Jones) !

The term *propositional function* thus vanishes, its work being done partly by forms and partly by variable instances of these forms.

§ 4. *Dyadic Relational Complexes and Double Descriptive Functions.*—There is an adumbration of the notion of relational complexes in *Principia*, vol. i., *38, where 'double descriptive functions' are dealt with. In a sense all the notation here to be proposed is based on this notion. But it is evident that Russell and Whitehead think that only a few relations give rise to such functions. Moreover, the notation there developed only applies to *double* descriptive functions. Now in geometry and in many other regions we need to deal with *multiple* descriptive functions.

My object now is to generalise this notion and apply it (i) to all dyadic relations including the relation ϵ of a member to its class, and (ii) to extend it to relations of all degrees of polyadicity. We will begin with dyadic relations.

I. Relational Complexes and their Associated Functions.

§ 5. *Notation for Dyadic Complexes.*—Let α be a class, and x a variable individual. Then

$\epsilon(x, \alpha)$ denotes x's membership of α.
$\epsilon(x, \alpha)\ !$ denotes $x\epsilon\alpha$.

It is of course clear from à priori considerations that α must be of a type above that of x, and again that $\epsilon(\alpha, x)$ is nonsense.

Let us now leave ϵ for the moment and consider any dyadic relation R whose terms, we will suppose, are individuals. *E.g.* let R = the relation of loving.

Then $R(x, y)$ is the love of x for y.[1]

Now what would $R(x, -)$ be? Let us define this as the relation of $R(x, y)$ to y. (*Cf.* $x\S y$ in *Principia*, when $\S y$ is the relation of $x\S y$ to x.)

Similarly $R(-, y)$ is the relation of $R(x, y)$ to x.

Now consider Russell's $x\S``\beta$. This is the class

$$\hat{u}[(\exists y) . y\epsilon\beta . u = x\S y].$$

I propose to denote this class by the symbol $R(x, ``\beta)$.

Similarly $R(``\alpha, y)$ will be Russell's $\S y``\alpha$.

E.g. $R(x, ``\beta)$ might be the class of x's love affairs with Frenchmen.

$R(``\alpha, y)$ might be the class of the love affairs of Englishmen for y.

Now $R(``\alpha, y)$ is symbolised by Russell not only as $\S y``\alpha$ but also as $\alpha \overset{\S}{,,} y$, and this is done in order that it may in its turn be treated as a double descriptive function. Our notation allows us to do likewise. We see at once that we can derive two new relations from our classes, *e.g.*, $R(``\alpha, -)$ from $R(``\alpha, y)$ and $R(-, ``\beta)$ from $R(x, ``\beta)$. The former might mean the relation of (the love affairs of Englishmen for y) to y, and the latter the relation of (the love affairs, of x with Frenchmen) to x. $R(``\alpha, -)$ is what Russell symbolises by $\alpha \overset{\S}{,,}$. His symbol for my $R(-, ``\beta)$ would presumably be $\overset{\S}{,,} \beta$.

§ 6. *Derivative Classes of Classes.*—From the relation

[1] Strictly there seems to be a difference between x's love for y, the fact that x loves y, and x's love-affair with y. It would be necessary in any complete treatment to analyse these carefully, and, if they proved to be genuinely different, to establish a different symbol for each. In the present tentative sketch I have treated them as equivalent, and in particular examples have translated $R(x, y)$ into the form of words that seemed most convenient in each case.

$R(-, ``\beta)$ we can get a new class, this time a class of classes. This will be symbolised by $R(``\underline{\alpha}, ``\beta)$. What will this mean?

We have
$$\gamma \epsilon R(``\underline{\alpha}, ``\beta) . \equiv : (\exists x) . x\epsilon\alpha . \gamma = R(x, ``\beta).$$
$$\equiv : (\exists x) . x\epsilon\alpha . \gamma = \hat{u}[(\exists y) . y\epsilon\beta . u = R(x, y)].$$

Hence
$$R(``\underline{\alpha}, ``\beta) = \hat{\gamma}[(\exists x) . x\epsilon\alpha . \gamma = \hat{u}[(\exists y) . y\epsilon\beta . u = R(x, y)]].$$

Now why do we write this in the form $R(``\underline{\alpha}, ``\beta)$ and not simply in the form $R(``\alpha, ``\beta)$? The reason is this. The relations $R(``\alpha, -)$ and $R(-, ``\beta)$ are different, and they give rise to different classes of classes. If we do not show which relation we started with we shall end up with $R(``\alpha, ``\beta)$ in both cases, *i.e.*, we shall have *one* symbol $R(``\alpha, ``\beta)$ to represent the two different classes

$$\hat{\gamma}[(\exists y) . y\epsilon\beta . \gamma = \hat{u}[(\exists x) . x\epsilon\alpha . u = R(x, y)]]$$
and
$$\hat{\gamma}[(\exists x) . x\epsilon\alpha . \gamma = \hat{u}[(\exists y) . y\epsilon\beta . u = R(x, y)]].$$

We must therefore have some means of distinguishing in the final symbol between the relation with which we started. Accordingly I propose to write

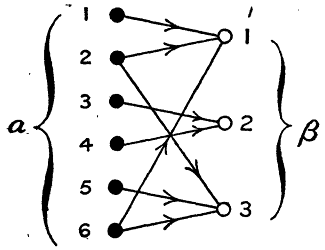
$R(``\alpha, ``\beta)$ for the class corresponding to $R(-, ``\beta)$
and $R(``\alpha, ``\underline{\beta})$ for the class corresponding to $R(``\alpha, -)$.

The class $R(``\underline{\alpha}, ``\beta)$ is Russell's class $\alpha \underset{\text{''}}{\S} ``\beta$.

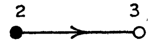
Now since $R(``\underline{\alpha}, ``\beta)$ and $R(``\alpha, ``\underline{\beta})$ are classes of classes they will have logical sums. And it is easy to prove the important proposition that
$$s`R(``\underline{\alpha}, ``\beta) = s`R(``\alpha, ``\underline{\beta}) = \hat{u}[(\exists x, y) . x\epsilon\alpha . y\epsilon\beta . u = R(x, y)].^{1}$$

We can easily illustrate all these notions by means of a diagram. Suppose, *e.g.*, that there are 6 Englishmen and 3 Frenchmen. Let us represent Englishmen by dots and Frenchmen by circles. Let us represent the love of the Englishman m for the Frenchman n by $\overset{m}{\bullet}\longrightarrow\overset{n}{\circ}$. Then we might have the following state of affairs:—

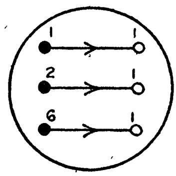
[1] Since these logical sums are important and do not depend on the difference between $R(``\underline{\alpha}, ``\beta)$ and $R(``\alpha, ``\underline{\beta})$ it will be useful to have a symbol for them. I suggest that $R(``\alpha, ``\overline{\beta})$ be used; it can hardly lead to error.



Then

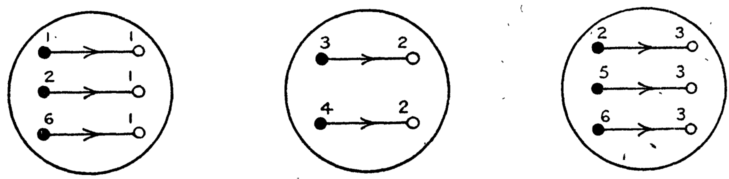
(i) 

represents the love of E_2 for F_3.

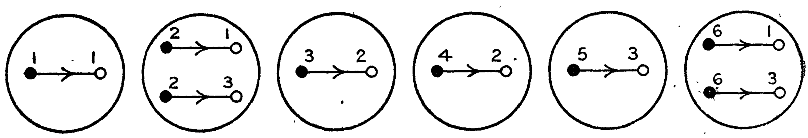
(ii) 

represents the class of loves of Englishmen for F_1, *i.e.*, the class $R(``a, F_1)$.

(iii) $R(``a, ``\underline{\beta})$ is the class whose members are the classes.



(iv) $R(``\underline{a}, ``\beta)$ is the class whose members are the classes.



We see in fact that $R(``\underline{a}, ``\beta)$ and $R(``a, ``\underline{\beta})$ are two different classifications of the loves of these Englishmen for these Frenchmen. The first classifies together all loves in which the same Englishman is the lover and the second classifies together all loves in which the same Frenchman is the beloved.

(v) It is clear from these diagrams that

$$s`R(``a, ``\underline{\beta}) = s`R(``\underline{a}, ``\beta),$$

and that it is the class of all the eight friendships in which an Englishman loves a Frenchman.

§ 7. *Application to* ϵ.—As these results hold generally of dyadic relations we can apply them at once to ϵ.

For example we shall have $\epsilon(``\gamma, a)$ as the class of memberships in $a$ of members of $\gamma$. Again $\epsilon(x, ``\kappa^2)$—where κ^2 is written to denote the fact that κ must be a class of classes—stands for the class of memberships of x in classes which are themselves members of κ. Lastly we shall have:

$$s`\epsilon(``\underline{\gamma}, ``\kappa^2) = s`\epsilon(``\gamma, ``\underline{\kappa}^2) = \hat{u}[\exists x, a) . x\epsilon\gamma . a\epsilon\kappa . u = \epsilon(x, a)].$$

This is thus the class of memberships of members of γ in classes that are members of κ. I do not suggest that in the case of ϵ these functions are of much practical importance. If we want an example from geometry we can take the relation between two segments h and k which make an angle with each other. Then $\angle(h, k)$! expresses the fact that h makes an angle with k and $\angle(h, k)$ represents the angle which h makes with k.

$\angle(h, \text{``}\beta)$ is the class of angles made by $h$ with segments of the class $\beta$; $\angle(\text{``}a, k)$ is the class of angles made by segments of the class a with k; $s\text{`}\angle(\text{``}a, \text{``}\beta)$ or $\angle(\text{``}a, \text{``}\beta)$ is the class of angles formed by a member of a with a member of β.

§ 8. *Extension to Triadic Relations.*—Conformably to what has been said above a triadic relational complex will have the form $R(-, -, -)$. Let R be the relation of jealousy. Then $R(x, y, z)$ is the jealousy of x for y on account of z; $R(-, y, z)$ is the relation of this jealousy to x, $R(x, -, z)$ is the relation of it to y, and $R(x, y, -)$ is the relation of it to z.

The next point to notice is that a formula such as $R(-, -, z)$ must be rejected as ambiguous on similar grounds to those which made us reject $R(\text{``}a, \text{``}\beta)$. For $R(-, -, z)$ would equally stand for the relation of $R(x, -, z)$ to x and for the relation of $R(-, y, z)$ to y; and these are clearly not identical with each other. If we want to express these relations we must do so by the respective formulæ $R(-, =, z)$, and $R(=, -, z)$. Clearly there will be six such relations, *viz.*,

$$
\begin{gathered}
R(=, -, z) \text{ and } R(-, =, z) \\
R(=, y, -) \text{ and } R(-, y, =) \\
R(x, =, -) \text{ and } R(x, -, =).
\end{gathered}
$$

We could evidently go a step further and consider the relations of each of these to the remaining term in it. Their symbols would be

$$
\begin{gathered}
R(\equiv, =, -) \text{ and } R(=, \equiv, -) \\
R(\equiv, -, =) \text{ and } R(=, -, \equiv) \\
R(-, \equiv, =) \text{ and } R(-, =, \equiv).
\end{gathered}
$$

I shall not attempt to translate these symbols into words. A simplification which suggests itself and which would clearly be useful in dealing with relations of higher degrees of polyadicity is shown below when the above six formulæ are written respectively as :—

$$\mathrm{R}(\overset{3}{-}, \overset{2}{-}, \overset{1}{-}) \text{ and } \mathrm{R}(\overset{2}{-}, \overset{3}{-}, \overset{1}{-})$$
$$\mathrm{R}(\overset{3}{-}, \overset{1}{-}, \overset{2}{-}) \text{ and } \mathrm{R}(\overset{2}{-}, \overset{1}{-}, \overset{3}{-})$$
$$\mathrm{R}(\overset{1}{-}, \overset{3}{-}, \overset{2}{-}) \text{ and } \mathrm{R}(\overset{1}{-}, \overset{2}{-}, \overset{3}{-}).$$

(Naturally with a dyadic relation we should get

$\mathrm{R}(\overset{1}{-}, \overset{2}{-})$ and $\mathrm{R}(\overset{2}{-}, \overset{1}{-})$ simply.)

§ 9. *Classes derived from Triadic Complexes.*—From the relations $\mathrm{R}(-, y, z)$, $\mathrm{R}(x, -, z)$, and $\mathrm{R}(x, y, -)$ we at once derive the classes $\mathrm{R}(``\alpha, y, z)$, $\mathrm{R}(x, ``\beta, z)$ and $\mathrm{R}(x, y, ``\gamma)$. These may be illustrated respectively by (i) The jealousies of Englishmen for y on account of z, (ii) The jealousies of x for Frenchmen on account of z, and (iii) The jealousies of x for y on account of Germans.

These classes give rise respectively to the relations

$$\mathrm{R}(``\alpha, -, z) \text{ and } \mathrm{R}(``\alpha, y, -)$$
$$\mathrm{R}(-, ``\beta, z) \text{ and } \mathrm{R}(x, ``\beta, -)$$
$$\mathrm{R}(-, y, ``\gamma) \text{ and } \mathrm{R}(x, -, ``\gamma).$$

From these we can obtain in the usual way six classes of classes of relational complexes, *viz.*,

$$\mathrm{R}(``\alpha, ``\underline{\beta}, z) \text{ and } \mathrm{R}(``\alpha, y, ``\underline{\gamma})$$
$$\mathrm{R}(``\underline{\alpha}, ``\beta, z) \text{ and } \mathrm{R}(x, ``\beta, ``\underline{\gamma})$$
$$\mathrm{R}(``\underline{\alpha}, y, ``\gamma) \text{ and } \mathrm{R}(x, ``\underline{\beta}, ``\gamma).$$

It will be sufficient to illustrate the meanings of the first and third of these.

$$\text{We have } \delta\epsilon\mathrm{R}(``\alpha, ``\underline{\beta}, z) \,.\equiv.\, (\exists y) \,.\, y\epsilon\beta \,.\, \delta = \mathrm{R}(``\alpha, y, z).$$
$$\equiv.\, (\exists y) \,.\, y\epsilon\beta \,.\, \delta = \hat{u}[(\exists x) \,.\, x\epsilon\alpha \,.\, u = \mathrm{R}(x, y, z)].$$
$$\text{Again} \quad \delta\epsilon\mathrm{R}(``\underline{\alpha}, ``\beta, z) \,.\equiv.\, (\exists x) \,.\, x\epsilon\alpha \,.\, \delta = \hat{u}[(\exists y) \,.\, y\epsilon\beta \,.\, u = \mathrm{R}(x, y, z)].$$

The first means that you first consider all the jealousies in which any Englishman is jealous of y on account of z, where y is a Frenchman, and then make up a class each of whose members is the class of these jealousies directed at a single Frenchman. The second means that you first consider all the jealousies in which x is jealous of any Frenchman on account of z, where x is an Englishman. You then make up a class each of whose members is the class of these jealousies felt by a single Englishman.

It is evident that $s`\mathrm{R}(``\underline{\alpha}, ``\beta, z) = s`\mathrm{R}(``\alpha, ``\underline{\beta}, z) = \hat{u}[(\exists x, y) \,.\, x\epsilon\alpha \,.\, y\epsilon\beta \,.\, u = \mathrm{R}(x, y, z)]$. It is thus the class of jealousies of Englishmen for Frenchmen on account of z.

§ 10. Our six new classes give rise to six new relations, *viz.*,

$$
\begin{gathered}
R(\text{“}\alpha, \text{“}\underline{\beta}, -) \text{ and } R(\text{“}\alpha, -, \text{“}\underline{\gamma}) \\
R(\text{“}\underline{\alpha}, \text{“}\beta, -) \text{ and } R(-, \text{“}\beta, \text{“}\underline{\gamma}) \\
R(\text{“}\underline{\alpha}, -, \text{“}\gamma) \text{ and } R(-, \text{“}\underline{\beta}, \text{“}\gamma)
\end{gathered}
$$

These in turn will give rise to six classes of classes of classes, *viz.*,

$$
\begin{gathered}
R(\text{“}\alpha, \text{“}\underline{\beta}, \text{“}\underline{\underline{\gamma}}) \text{ and } R(\text{“}\alpha, \text{“}\underline{\underline{\beta}}, \text{“}\underline{\gamma}) \\
R(\text{“}\underline{\alpha}, \text{“}\beta, \text{“}\underline{\underline{\gamma}}) \text{ and } R(\text{“}\underline{\underline{\alpha}}, \text{“}\beta, \text{“}\underline{\gamma}) \\
R(\text{“}\underline{\alpha}, \text{“}\underline{\underline{\beta}}, \text{“}\gamma) \text{ and } R(\text{“}\underline{\underline{\alpha}}, \text{“}\underline{\beta}, \text{“}\gamma).
\end{gathered}
$$

Let us take the first and last of these as examples. It is easy to show that

$$
\begin{aligned}
R(\text{“}\alpha, \text{“}\underline{\beta}, \text{“}\underline{\underline{\gamma}}) = \hat{\kappa}[[[(\exists z) \,.\, z\epsilon\gamma \,.\, \kappa = \hat{\delta}[[(\exists y) \,.\, y\epsilon\beta \,.\, \delta \\
= \hat{u}[(\exists x) \,.\, x\epsilon\alpha \,.\, u = R(x, y, z)]]]
\end{aligned}
$$

and that

$$
\begin{aligned}
R(\text{“}\underline{\underline{\alpha}}, \text{“}\beta, \text{“}\underline{\gamma}) = \hat{\kappa}[[[(\exists x) \,.\, x\epsilon\alpha \,.\, \kappa = \hat{\delta}[[(\exists y) \,.\, y\epsilon\beta \,.\, \delta \\
= \hat{u}[(\exists z) \,.\, z\epsilon\gamma \,.\, u = R(x, y, z)]]].
\end{aligned}
$$

The interpretation of these classes in words would be intolerably tedious and would add nothing to the intelligibility of the notions. But the logical sum of the logical sum of these classes is important.[1]

It is in fact easy to prove that

$$
\begin{aligned}
s\text{‘}s\text{‘}R(\text{“}\alpha, \text{“}\underline{\beta}, \text{“}\underline{\underline{\gamma}}) &= s\text{‘}s\text{‘}R(\text{“}\alpha, \text{“}\underline{\underline{\beta}}, \text{“}\underline{\gamma}) \\
&= s\text{‘}s\text{‘}R(\text{“}\underline{\alpha}, \text{“}\beta, \text{“}\underline{\underline{\gamma}}) = \ldots \\
&= \hat{u}[(\exists x, y, z) \,.\, x\epsilon\alpha \,.\, y\epsilon\beta \,.\, z\epsilon\gamma \,.\, u = R(x, y, z)].
\end{aligned}
$$

Interpreting this class in words we see that it is the class of jealousies felt by Englishmen for Frenchmen on account of Germans.

§ 11. *Further Extension of Dyadic Complexes.*—We may say that so far we have dealt with classes of complexes obtained from a single relation R by varying the terms within the limits of certain classes $\alpha, \beta, \ldots$ But we might keep the terms constant and vary the relation within a certain class ρ of relations, which must, for our purpose, be assumed only to contain relations of the same polyadicity.

[1] This may conveniently and without risk of error be represented by the otherwise meaningless formula $R(\text{“}\alpha, \text{“}\beta, \text{“}\gamma)$.

E.g. let ρ be a class of dyadic relations. Consider the class

$$\delta = \hat{u}[(\exists \mathrm{R}) . \mathrm{R}\epsilon\rho . u = \mathrm{R}(x, y)].$$

Let us define a new relation $-(x, y)$ as follows:—

$$-(x, y) = \hat{u}\hat{\mathrm{R}}[u = \mathrm{R}(x, y)] \text{ Df.}$$

Then, in Russell's notation, $\delta = [-(x, y)]``\rho$

$= \text{in our notation, } ``\rho(x, y).$

We can now proceed to generalise this further by varying x and y.

Clearly $``\rho(-, y)$ is the relation of $``\rho(x, y)$ to x.

Hence

$$``\rho(``\underline{a}, y) = \hat{\gamma}[[(\exists x) . x\epsilon a . \gamma = \hat{u}[(\exists \mathrm{R}) . \mathrm{R}\epsilon\rho . u = \mathrm{R}(x, y)]].$$

Whence $s`[``\rho(``\underline{a}, y)] = \hat{u}[(\exists x, \mathrm{R}) . x\epsilon a . \mathrm{R}\epsilon\rho . u = \mathrm{R}(x, y)]$

whilst $s`[``\rho(x, ``\underline{\beta})] = \hat{u}[(\exists y, \mathrm{R}) . y\epsilon\beta . \mathrm{R}\epsilon\rho . u = \mathrm{R}(x, y)].$

We must now notice another relation and another class which must not be confused with the foregoing ones. Taking the class $\mathrm{R}(``a, y)$ we can form the relation $-(``a, y)$, which is that of $\mathrm{R}(``a, y)$ to R.

From this relation we can get the class of classes $``\underline{\rho}(``a, y)$. Now it is easy to see that

$$``\underline{\rho}(``a, y) = \hat{\gamma}[[(\exists \mathrm{R}) . \mathrm{R}\epsilon\rho . \gamma = \hat{u}[(\exists x) . x\epsilon a . u = \mathrm{R}(x, y)]].$$

Similarly

$$``\underline{\rho}(x, ``\beta) = \hat{\gamma}[[(\exists \mathrm{R}) . \mathrm{R}\epsilon\rho . \gamma = \hat{u}[(\exists y) . y\epsilon\beta . u = \mathrm{R}(x, y)]].$$

It is evident that $s`[``\underline{\rho}(``a, y)] = s`[``\rho(``\underline{a}, y)]$

and that $s`[``\underline{\rho}(x, ``\beta)] = s`[``\rho(x, ``\underline{\beta})]$.

§ 12. We can now consider some new classes of classes of classes.

(i) $\mathrm{R}(``\underline{a}, ``\beta)$ produces the relation $-(``\underline{a}, ``\beta)$ between it and R, and the class $``\underline{\rho}(``\underline{a}, ``\beta)$.

(ii) $\mathrm{R}(``a, ``\underline{\beta}$ produces the relation $-(``a, ``\underline{\beta})$ between it and R, and the class $``\underline{\rho}(``a, ``\underline{\underline{\beta}})$.

(iii) $``\underline{\rho}(``a, y)$ produces the relation $``\underline{\rho}(``a, -)$ between it and y, and the class $``\underline{\rho}(``a, ``\underline{\underline{\beta}})$.

(iv) $``\rho(``\underline{a}, y)$ produces the relation $``\rho(``\underline{a}, -)$ between it and y, and the class $``\rho(``\underline{a}, ``\underline{\underline{\beta}})$.

(v) $``\underline{\rho}(x, ``\beta)$ produces the relation $``\underline{\rho}(-, ``\beta)$ between it and x, and the class $``\underline{\rho}(``\underline{\underline{a}}, ``\beta$.

(vi) $``\rho(x, ``\underline{\beta})$ produces the relation $``\rho(-, ``\underline{\beta})$ between it and x, and the class $``\rho(``\underline{\underline{a}}, ``\underline{\underline{\beta}})$.

There are six of these classes in all as with ordinary triadic complexes like R(x, y, z). Of these we will consider (i), (iii), and (vi), which illustrate ρ in different states.

$$\text{(i)}\ \text{“}\underline{\rho}(\text{“}\underline{\alpha}, \text{“}\beta) = \hat{\kappa}[[[(\exists \mathrm{R}) . \mathrm{R}\epsilon\rho . \kappa = \hat{\gamma}[[(\exists x) . x\epsilon\alpha . \gamma = \hat{u}[(\exists y) . y\epsilon\beta . u = \mathrm{R}(x, y)]]].$$

$$\text{(iii)}\ \text{“}\underline{\rho}(\text{“}\alpha, \text{“}\underline{\beta}) = \hat{\kappa}[[[(\exists y) . y\epsilon\beta . \kappa = \hat{\gamma}[[(\exists \mathrm{R}) . \mathrm{R}\epsilon\rho . \gamma = \hat{u}[(\exists x) . x\epsilon\alpha . u = \mathrm{R}(x, y)]]].$$

$$\text{(vi)}\ \text{“}\rho(\text{“}\underline{\alpha}, \text{“}\underline{\beta}) = \hat{\kappa}[[[(\exists x) . x\epsilon\alpha . \kappa = \hat{\gamma}[[(\exists y) . y\epsilon\beta . \gamma = \hat{u}[(\exists \mathrm{R}) . \mathrm{R}\epsilon\rho . u = \mathrm{R}(x, y)]]].$$

It is evident that s‘s‘ [any of these classes] is the same. It may be represented according to our usual convention by “ρ(“α, “β). We then have

$$\text{“}\rho(\text{“}\alpha, \text{“}\beta) = \hat{u}[\exists x, y, \mathrm{R}) . x\epsilon\alpha . y\epsilon\beta . \mathrm{R}\epsilon\rho . u = \mathrm{R}(x, y)].$$

Suppose, *e.g.*, that ρ was the class of rectilinear relations and that, when R$\epsilon\rho$, R(x, y) represents the segment on the line R which is terminated by the points x and y.

Let α and β be two planes. Then “ρ(“α, “β) is the class of segments of each of which one end is on the plane α and the other end is on the plane β.

Evidently this extension could be applied to triadic and higher relational complexes. But there is no need for us to trouble about this, for enough has been given to show that we have a general notation capable of being applied consistently to relational complexes of any degree of polyadicity.

II. Relational Propositions and their Associated Functions.

§ 13. *Definition of the Present Problem.*—We are now going to consider the extension of such notions as Russell denotes by R‘y, R“β, $\overrightarrow{\mathrm{R}}$‘$y$, D‘R, and $\breve{\mathrm{R}}$. We shall try to establish a system of notation which will (*a*) apply consistently to relations of all degrees of polyadicity; and (*b*) show as much connexion as possible with that already developed above for complexes and their associated functions.

We must remember that our previous notation has applied mainly, not to R or to terms in R's field, but to relational complexes, such as R(x, y, z), and to classes of these. It is perfectly true that, *in connexion with such complexes*, we have considered special cases of the general notion R“β. *E.g.*, we have considered the class R$(x$, “β, $z)$. But the relations with which we then dealt were always of one special kind, *viz*. the relations of complexes to some of their own terms, *e.g.*, the relation R$(x, -, z)$. Now, although all relations give rise to complexes and hence to relations between these complexes and their terms, it is of course not true that all relations relate complexes to their terms. Most relations relate terms within a complex to each other. Hence a notation which is convenient for relations of the special kind which we have been considering so far will not necessarily be convenient or even possible for relations in general.

We may begin by noticing the following important connexion between relational complexes and relational propositions:—

$$\mathrm{R}(x, y, z)! \,.\equiv.\, \mathrm{E!R}(x, y, z) \,.\equiv.\, (\exists u) \,.\, u = \mathrm{R}(x, y, z),$$

e.g., x is jealous of y on account of z . ≡ . the jealousy of x for y on account of z exists . ≡ . there is something which is identical with x's jealousy for y on account of z.

§ 14. *Extension of* $\overrightarrow{\mathrm{R}}$.—$\overrightarrow{\mathrm{R}}$'$y$ is defined as $\hat{x}[x\mathrm{R}y]$, whilst $\overleftarrow{\mathrm{R}}$'$x$ is defined as $\hat{y}[x\mathrm{R}y]$.

Now Russell's $x\mathrm{R}y$ is our $\mathrm{R}(x, y)!$.

So $\overrightarrow{\mathrm{R}}\text{'}y = \hat{x}[\mathrm{R}(x, y)!]$.

Let us denote this class by the symbol $\mathrm{R}(\rightarrow, y)$. Then Russell's $\overrightarrow{\mathrm{R}}$ is the relation between $\mathrm{R}(\rightarrow, y)$ and y, which in our system of notation is written $\mathrm{R}(\rightarrow, -)$.

Similarly we shall write Russell's $\overleftarrow{\mathrm{R}}$ as $\mathrm{R}(-, \rightarrow)$ and his $\overleftarrow{\mathrm{R}}$'$x$ as $\mathrm{R}(x, \rightarrow)$.

It is now easy to extend the notation to triadic relations. Taking the proposition $\mathrm{R}(x, y, z)!$ we shall get the classes

(i) $\hat{x}[\mathrm{R}(x, y, z)!] = \mathrm{R}(\rightarrow, y, z)$, *e.g.*, those who are jealous of y on account of z.

(ii) $\hat{y}[\mathrm{R}(x, y, z)!] = \mathrm{R}(x, \rightarrow, z)$, *e.g.*, those of whom x is jealous on account of z.

(iii) $\hat{z}[\mathrm{R}(x, y, z)!] = \mathrm{R}(x, y, \rightarrow)$, *e.g.*, those on whose account x is jealous of y.

Now each of these will give rise at once to the relations

$$\begin{array}{l} \mathrm{R}(\rightarrow, -, z) \text{ and } \mathrm{R}(\rightarrow, y, -) \\ \mathrm{R}(-, \rightarrow, z) \text{ and } \mathrm{R}(x, \rightarrow, -) \\ \mathrm{R}(-, y, \rightarrow) \text{ and } \mathrm{R}(x, -, \rightarrow). \end{array}$$

These in the usual way will give rise to classes of classes. To see what these will be let us take, *e.g.*,

$$\mathrm{R}(\rightarrow, \text{“}\beta, z) \text{ and } \mathrm{R}(\text{“}\alpha, \rightarrow, z).$$

Then it is easy to see that

$$R(\rightarrow, \text{``}\beta, z) = \hat{\gamma}[(\exists y) \,.\, y \epsilon \beta \,.\, \gamma = R(\rightarrow, y, z)]$$

and that $R(\text{``}\alpha, \rightarrow, z) = \hat{\gamma}[(\exists x) \,.\, x \epsilon \alpha \,.\, \gamma = R(x, \rightarrow, z)]$.

E.g., $R(\rightarrow, \text{``}\beta, z)$ might be the class whose members are classes of persons who are jealous of some Frenchman on account of $z$; whilst $R(\text{``}\alpha, \rightarrow, z)$ might be the class whose members are the classes of persons of whom some Englishman is jealous on account of z.

It can easily be shown that

$$s\text{`}R(\rightarrow, \text{``}\beta, z) = \hat{x}[(\exists y) \,.\, y \epsilon \beta \,.\, R(x, y, z)!]$$

whilst $s\text{`}R(\text{``}\alpha, \rightarrow, z) = \hat{y}[(\exists x) \,.\, x \epsilon \alpha \,.\, R(x, y, z)!]$.

These might be respectively the class of persons each of whom is jealous of some Frenchman on account of z, and the class of persons of each of whom some Englishman is jealous on account of z.

§ 15. From the (class)[2] $R(\rightarrow, \text{``}\beta, z)$ we can as usual get the relation $R(\rightarrow, \text{``}\beta, -)$. And from this, as usual, we can get the (class)[3] $R(\rightarrow, \text{``}\beta, \text{``}\gamma)$. Similarly from $R(\text{``}\alpha, \rightarrow, z)$ we can get the (class)[3] $R(\text{``}\alpha, \rightarrow, \text{``}\gamma)$. Six such classes are possible with a triadic relation, *viz.*,

$$R(\leftarrow, \text{``}\beta, \text{``}\gamma) \text{ and } R(\rightarrow, \text{``}\beta, \text{``}\gamma)$$
$$R(\text{``}\alpha, \rightarrow, \text{``}\gamma) \text{ and } R(\text{``}\alpha, \rightarrow, \text{``}\gamma)$$
$$R(\text{``}\alpha, \text{``}\beta, \Rightarrow) \text{ and } R(\text{``}\alpha, \text{``}\beta, \rightarrow).$$

It can be shown without difficulty that

$$s\text{`}s\text{`}R(\rightarrow, \text{``}\beta, \text{``}\gamma) = s\text{`}s\text{`}R(\rightarrow, \text{``}\beta, \text{``}\gamma) = \hat{x}[(\exists y, z) \,.\, y \epsilon \beta \,.\, z \epsilon \gamma \,.\, R(x, y, z)!$$

Similarly we can show that the logical sum of the logical sum of the other two corresponding pairs is respectively

$$\hat{y}[(\exists x, z) \,.\, x \epsilon \alpha \,.\, z \epsilon \gamma \,.\, R(x, y, z)!] \text{ and } \hat{z}[(\exists x, y) \,.\, x \epsilon \alpha \,.\, y \epsilon \beta \,.\, R(x, y, z)!].$$

As an illustration, $s\text{`}s\text{`}R(\rightarrow, \text{``}\beta, \text{``}\gamma)$ might be the class of persons who are jealous of some Frenchman on account of some German.

§ 16. *Extension of* R``.—We are now in a position to deal with such notions as $R\text{``}\beta$. Let us begin with dyadic relations and then extend our results to relations of higher polyadicity. If R be a dyadic relation $R\text{``}\beta$ is defined as

$$\hat{x}[(\exists y) \,.\, y \epsilon \beta \,.\, R(x, y)!].$$

Evidently we must not use the notation $R(x, \text{``}\beta)$ for this class. For we have already used it to denote a class of *relational complexes, viz.*, $\hat{u}[(\exists y) \,.\, y\epsilon\beta \,.\, u = R(x, y)]$. But what we now want to symbolise is a class of *terms in a relational complex.*

Now, there is a close and interesting relation between $R``\beta$ and $R(x, ``\beta)$.

Remember that $R(x, y)! \,. \equiv, (\exists u) \,.\, u = R(x, y)$.

Then $$\begin{aligned} R``\beta &= \hat{x}[(\exists u) : (\exists y) \,.\, y\epsilon\beta \,.\, u = R(x, y)] \\ &= \hat{x}[(\exists u) \,.\, u\epsilon R(x, ``\beta)] \\ &= \hat{x}[E!R(x, ``\beta)]. \end{aligned}$$

Now I suggest that the class $\hat{x}[\exists!R(x, ``\beta)$ should be symbolised by the formula $R(!, ``\beta)$. Hence for Russell's $R``\beta$ we shall write $R(!, ``\beta)$.

Now consider the class

$$\times\ \hat{y}[(\exists x) \,.\, x\epsilon\alpha \,.\, xRy].$$

This $$\begin{aligned} &= \hat{y}[(\exists x) \,.\, x\epsilon\alpha \,.\, R(x, y)!] \\ &= \hat{y}[(\exists u) : (\exists x) \,.\, x\epsilon\alpha \,.\, u = R(x, y)] \\ &= \hat{y}[\exists!R(``\alpha, y)]. \end{aligned}$$

This can be consistently symbolised as $R(``\alpha, !)$.

We have then a notation which is (*a*) readily extensible to relations of higher degrees of polyadicity, and (*b*) brings out forcibly the difference between $R``\beta$—a class defined by relational propositions—and $R(x, ``\beta)$—a class whose members are relational complexes.

We must carefully note that, in spite of the appearance to the contrary, we cannot pass back from $R(``\alpha, !)$ to a relational complex $R(x, !)$ and suppose that the class $R(``\alpha, !)$ is generated from the complex $R(x, !)$ by a relation $R(-, !)$ between the complex and x. The fact is that whenever we are given a complex containing an individual or a class as a term we can go on to derive a relation between it and that individual or class. And from this we can construct a class of such complexes by substituting for the individual a class with two commas or for the class a (class)^2 with two commas. This we have already done with $R(\rightarrow, y)$. But when we start with a class of the form $R(!, ``\beta)$ we cannot assume that the opposite path can be trodden and that $R(!, ``\beta)$ *must* have been derived from a complex such as $R(!, y)$ through a relation $R(!, -)$. Under the present circumstances we are precluded from using the formula $R(x, !)$ or $R(!, y)$ for any purpose whatever. For if we could use it we could derive from it $R(``\alpha, !)$ and $R(!, ``\beta)$ respectively according to the general rules of our notation. But these have already had a meaning assigned to them, and it is such that they *cannot* have been so derived. For, if they had been so derived, they would be classes *of* complexes or *of* classes, whereas they are classes of terms *in* complexes, and in most cases these terms are not themselves either complexes or classes.

§ 17. We must now remark that it can easily be proved that

$$s\text{‘}R(\rightarrow, \text{“}\beta) = R(\,!, \text{“}\beta)$$

and $$s\text{‘}R(\text{“}a, \rightarrow) = R(\text{“}a, !).$$

With this preliminary proposition we can proceed to extend the notion of R“ to triadic relations. Starting with $R(x, y, z)$ we can get the following classes :—

$$R(\text{“}a, !, z) \text{ and } R(\text{“}a, y, !)$$
$$R(x, \text{“}\beta, !) \text{ and } R(!, \text{“}\beta, z)$$
$$R(x, !, \text{“}\gamma) \text{ and } R(!, y, \text{“}\gamma).$$

Then $R(\text{“}a, !, z) = \hat{y}[(\exists x) . x \epsilon a . R(x, y, z)!]$ with corresponding meanings for the others. We see that

$R(\text{“}a, !, z) = s\text{‘}R(\text{“}a, \rightarrow, z)$, and similarly for the others. *E.g.*, $R(\text{“}a, !, z)$ might mean the class of people of whom some Englishmen are jealous on account of z.

Now each of these classes will give rise to a relation between itself and the remaining individual in it. These relations give rise to six classes of classes, *viz.*,

$$R(\text{“}a, !, \text{“}\underline{\gamma}) \text{ and } R(\text{“}a, \text{“}\underline{\beta}, !)$$
$$R(\text{“}a, \text{“}\overline{\beta}, !) \text{ and } R(!, \text{“}\overline{\beta}, \text{“}\underline{\gamma})$$
$$R(\text{“}\underline{a}, !, \text{“}\gamma) \text{ and } R(!, \text{“}\underline{\beta}, \text{“}\gamma).$$

Now, *e.g.*, $R(\text{“}a, !, \text{“}\underline{\gamma}) = \hat{\delta}[(\exists z) . z \epsilon \gamma . \delta = R(\text{“}a, !, z)]$ and $R(\text{“}\underline{a}, !, \text{“}\gamma) = \hat{\delta}[(\exists x) . x \epsilon a . \delta = R(x, !, \text{“}\gamma)]$.

It is easy to prove from this that

$$s\text{‘}R(\text{“}a, !, \text{“}\underline{\gamma}) = s\text{‘}s\text{‘}R(\text{“}a, \rightarrow, \text{“}\underline{\gamma}) = \hat{y}[(\exists z, x) . z \epsilon \gamma . x \epsilon a . R(x, y, z)!].$$

Now it will be useful to have a simpler notation for such classes as $s\text{‘}R(\text{“}a, !, \text{“}\underline{\gamma})$ or $s\text{‘}s\text{‘}R(\text{“}a, \rightarrow, \text{“}\underline{\gamma})$. I suggest that they should be denoted by the symbol $R(\text{“}a, !!, \text{“}\gamma)$, etc. An obvious further simplification which will be useful in dealing with relations of higher polyadicity is to write $!^2$ for !!. We shall thus get three important classes, *viz.*,

$$R(!^2, \text{“}\beta, \text{“}\gamma) = \hat{x}[(\exists y, z) . y \epsilon \beta . z \epsilon \gamma . R(x, y, z)!]$$
$$R(\text{“}a, !^2, \text{“}\gamma) = \hat{y}[(\exists z, x) . z \epsilon \gamma . x \epsilon a . R(x, y, z)!]$$

and $$R(\text{“}a, \text{“}\beta, !^2) = \hat{z}[(\exists x, y) . x \epsilon a . y \epsilon \beta . R(x, y, z)!].$$

E.g., the first of these might be the class of people who are jealous of some Frenchmen on account of some Germans.

§ 18. We have thus found that logical *sums* of certain

$\rightarrow$ classes are important in the case of dyadic relations, and sums of sums of similar classes in that of triadic relations. This naturally leads us to inquire whether the logical *products* of the same classes might not be of sufficient importance to deserve a special symbolism.

Let us consider p'R($\rightarrow$, "β). It is easy to show that

$$p\text{'R}(\rightarrow, \text{"}\beta) = \hat{x}[y\epsilon\beta\,)_y\,\text{R}(x, y)!],$$

and that $$p\text{'R}(\text{"}a, \rightarrow) = \hat{y}[x\epsilon a\,)_x\,\text{R}(x, y)!].$$

We have denoted s'R($\rightarrow$, "β) by R(!, "β). Let us denote the corresponding product by substituting ¡ (a note of exclamation—or 'shriek' as Whitehead would call it—upside down) for !. We shall thus get the two classes

$$\text{R}(\text{"}a, ¡) \text{ and } \text{R}(¡, \text{"}\beta).$$

Now suppose we know that a (R (¡, "β). This means that $x\epsilon a\,.\,y\epsilon\beta)_{x,\,y}$ R(x, y)! The knowledge that β (R ("a, ¡) gives us the same information. Now this is often an important fact to symbolise. Suppose, *e.g.*, that β is the interior of a plane angle, and that R(x, y)! means that x can be joined to y by a segment that does not cut the sides of this angle. Then β (R (!, "β) would express the fact that any two points within the angle can be joined by a segment that does not cut the sides of the angle.

Another important piece of information can be symbolised by the statement $\exists$!$a \cap$R(!, "β). This tells us that there is at least one point in a and one in β which have to each other the relation R. Now these two statements may be regarded as defining two important relations, connected with R, between two classes. These relations might be symbolised respectively by R_p and R_s. Then

$$\text{R}_p = \hat{a}\hat{\beta}[x\epsilon a\,.\,y\epsilon\beta\,)_{x,\,y}\,.\,\text{R}(x, y)!] \text{ Df.}$$

and $$\text{R}_s = \hat{a}\hat{\beta}[\exists x, y)\,.\,x\epsilon a\,.\,y\epsilon\beta\,.\,\text{R}(x, y)!]. \text{ Df.}$$

§ 19. We can now go on to apply the same principles to triadic relations. We have so far considered only such classes as R(!!, "β, "γ), *i.e.*, s's'R($\rightarrow$, "β, "γ). But we could evidently consider three other classes obtained from

$$\text{R}(\rightarrow, \text{"}\beta, \text{"}\underline{\gamma}), \textit{ viz.},$$

p's'R($\rightarrow$, "β, "γ) which might be written R(¡!, "β"γ)
p'p'R($\rightarrow$, "β, "$\underline{\gamma}$) „ „ R)¡¡, "β, "γ)
and
s'p'R($\rightarrow$, "β, "$\underline{\gamma}$) „ „ R(!¡, "β, "γ).

Of these classes only one, so far as I have been able to see, is likely to be of great logical importance. This is R(¡!, β", "γ).

It can be shown without much difficulty that

$$R(!\,;,\ ``\beta,\ ``\gamma) = \hat{x}[y\epsilon\beta\,.\,z\epsilon y\,)_{y,z}\,.\,R(x,\ y,\ z)!].$$

This class derived from a triadic relation therefore corresponds to $R(;,\ ``\beta)$ derived from a dyadic relation.

Clearly $\alpha(R(;!,\ ``\beta,\ ``\gamma)\,.\equiv:x\epsilon\alpha\,.\,y\epsilon\beta\,.\,z\epsilon\gamma\,)_{x,y,z}\,.\,R(x,\ y,\ z)!.$

We thus have a derived triadic relation between α, β, γ which we can denote by R_{ps}, so that

$$R_{ps}(\alpha,\ \beta,\ \gamma)!\,.\equiv.\ \alpha(R\,(;!,\ ``\beta,\ ``\gamma).$$

The derived relation, obtained from a triadic R and comparable to R_s from a dyadic R, may be symbolised by R_{ss}.

$$R_{ss}(\alpha,\ \beta,\ \gamma)!\,.\equiv.\ (\exists x,\ y,\ z)\,.\,x\epsilon\alpha\,.\,y\epsilon\beta\,.\,z\epsilon\gamma\,.\,R(x,\ y,\ z)!\,.$$
$$\equiv.\ \exists!\alpha\cap R(!!,\ ``\beta,\ ``\gamma).$$

§ 20. *Geometrical Illustration.*—It may be of interest at this point to illustrate our notation by a geometrical example. For this purpose I shall translate the axioms on the relation of between in Hilbert's *Foundations of Geometry* (Eng. Trans., p. 6) into our notation.

Let π stand for the class of points, and λ for the class of rectilinear relations. Then the statement $\alpha\epsilon\mathrm{Cl}`\pi\cap s`\mathrm{Cl}``\mathrm{C}``\lambda$ will mean α is a class of collinear points. With these preliminary pieces of notation settled we can begin to deal with the relation of between. Let $T(x,\ y,\ z)!$ denote x is between y and z.

Then $T(x,\ y,\ z)!\,.\,)\,.\,\iota`x\cup\iota`y\cup\iota`z\epsilon\mathrm{Cl}`\pi\cap s`\mathrm{Cl}``\mathrm{C}``\lambda\cap 3.$

Now for Hilbert's axioms:—

(1) 'If A, B, and C are points of a straight line, and B lies between A and C, then B lies also between C and A.'

Translation.—$T(\rightarrow,\ y,\ z)\ (T(\rightarrow,\ z,\ y).$

(2) 'If A and C are two points of a straight line then there exists at least one point B lying between A and C, and at least one point D so situated that C lies between A and D.'

Translation.—$y, z\epsilon\pi\,.\,y \neq z\,.)\,.\ \exists!T(\rightarrow, y, z)\,.\ \exists!T(z, y, \rightarrow).$

(3) 'Of any three points situated on a straight line there is always one and only one between the other two.'

Translation.—$\alpha\epsilon\mathrm{Cl}`\pi\cap s`\mathrm{Cl}``\mathrm{C}``\lambda\cap 3\,.)\,.\ T(!!,\ ``\alpha,\ ``\alpha)\cap\alpha\epsilon 1.$

[1] (4) 'Any four points A, B, C, D of a straight line can always be arranged so that B shall lie between A and C and also between A and D, and, furthermore, so that C shall lie between A and D and also between B and D.'

I must remark in the first place that this axiom is very badly stated. You cannot *arrange* points on a line; they are in the order in which they are, and there is an end of the

[1] The 'axiom' has since been deduced from Hilbert's other axioms.

matter. What you can arrange is the letters by which you shall denote them. But an axiom can hardly deal with typographical matters like this. I shall therefore substitute for Hilbert's axiom the following, which, when combined with (3) seems to give all the necessary properties of linear order:—

$$\alpha\epsilon \mathrm{Cl}`\pi\cap s`\mathrm{Cl}``\mathrm{C}``\lambda\cap 4\,.\,)\,:\,\alpha\cap\mathrm{T}(!!,\,``\alpha,\,``\alpha)\epsilon 2\,:\,\alpha\cap\mathrm{T}(!!,\,``\alpha,\,``\alpha)$$
$$(\;\mathrm{T}\{!!,\,``\alpha\cap\mathrm{T}(!!,\,``\alpha,\,``\alpha),\,``\alpha-\mathrm{T}(!!,\,``\alpha,\,``\alpha)\}$$
$$\cap\mathrm{T}\{!!,\,``\alpha-\mathrm{T}(!!,\,``\alpha,\,``\alpha),\,``\alpha-\mathrm{T}(!!,\,``\alpha,\,``\alpha)\}.$$

This formidable looking proposition asserts that if α be a class of four collinear points then the members of α which are between members of α are two in number. Moreover, the members of α which are between members of α are between a member of α which is and a member of α which is not itself between members of α. Furthermore, the members of α which are between members of α are also between members of α which are not between members of α.

This is a fairly complex statement, and our notation expresses it with reasonable simplicity.

§ 21. *Extension of* D, Ɑ, *and* C.—If R be a dyadic relation D‘R is defined as $\hat{x}[(\exists y)\,.\,\mathrm{R}(x,y)!]$ and Ɑ‘R is defined as $\hat{y}[(\exists x)\,.\,\mathrm{R}(x,y)!]$.

Now consider the class R(!, “V) when V is the universe of entities of the type of y in $\mathrm{R}(x, y)$.

$$\mathrm{R}(!,\,``\mathrm{V}) = \hat{x}[(\exists y)\,.\,y\epsilon\mathrm{V}\,\dot{.}\,\mathrm{R}(x,y)!].$$

But $\qquad y\epsilon\mathrm{V}\,.\,\mathrm{R}(x,y)! \equiv \mathrm{R}(x,y)!$

Hence $\qquad \mathrm{D}`\mathrm{R} = \mathrm{R}(!,\,``\mathrm{V}).$

Similarly $\qquad$ Ɑ‘R = R(“V, !).

Hence D, on our notation, is – (!, “V) and Ɑ is – (“V, !) So D“λ becomes “λ(!, “V) and Ɑ“λ becomes “λ(“V, !). Now C‘R is defined as D‘R∪Ɑ‘R. Hence for us

$$\mathrm{C}`\mathrm{R} = \mathrm{R}(!,\,``\mathrm{V})\cup\mathrm{R}(``\mathrm{V},\,!).$$

It is easy to extend these results to relations of higher degrees of polyadicity. Here, however, the notion of domain and co-domain breaks down; it is better to say that there are as many different domains as there are degrees of polyadicity in the relation. Suppose we have a triadic R. Then we can denote its three domains by $\mathrm{D}_1`\mathrm{R}$, $\mathrm{D}_2`\mathrm{R}$, and $\mathrm{D}_3`\mathrm{R}$.

Then $\qquad \mathrm{D}_1`\mathrm{R} = \mathrm{R}(!!,\,``\mathrm{V},\,``\mathrm{V})$

$\qquad\qquad \mathrm{D}_2`\mathrm{R} = \mathrm{R}(``\mathrm{V},\,!!,\,``\mathrm{V})$

and $\qquad \mathrm{D}_3`\mathrm{R} = \mathrm{R}(``\mathrm{V},\,``\mathrm{V},\,!!).$

D_1, D_2, and D_3 will be the corresponding formulæ with – written for R. Naturally

$$\mathrm{C}`\mathrm{R} = \mathrm{R}(!!,\,``\mathrm{V},\,``\mathrm{V})\cup\mathrm{R}(``\mathrm{V},\,!!,\,``\mathrm{V})\cup\mathrm{R}(``\mathrm{V},\,``\mathrm{V},\,!!).$$

It might prove convenient, and could do no harm, to denote $C'R$ by $R(!, !)$ for dyadic relations and by $R(!^2, !^2, !^2)$ for triadic ones. But I should suppose that here, and indeed in all cases where we have dyadic relations whose dyadicity is guaranteed by logic itself and not merely postulated in the axioms of some special science with which we are dealing, the old Russell-Whitehead notation should be conserved with the slight modifications that I have suggested about domains. Hence, although we have shown that $R(!!, ``V, ``V)$ is the proper and consistent way to express $D_1'R$ on our notation, it would be pedantic not to use the shorter and more convenient $D_1'R$. The same remarks apply to such purely logical dyadic relations as s, ρ, Cl, etc., which nearly always occur in actual life in descriptive functions, and which are known by every one to be dyadic.

§ 22. *Extension of $R'y$.*—It remains for us to give a consistent symbolism for the notion $R'y$, *i.e.*, the term which has the relation R to y. Now here we are met by a problem somewhat similar to that which faced us in dealing with $R``\beta$. We then needed to symbolise a class of *terms* instead of a class of complexes; we succeeded in doing this by means of the connexion between $R``\beta$ and $s'\overrightarrow{R}``\beta$. Here we want to symbolise the *term* which has the relation R to y. Now, in particular cases, we have been able to do this with ease. *E.g.*, we have symbolised the *class* which has the relation $\overrightarrow{R}$ to y by $R(\rightarrow, y)$, and we have constantly symbolised the *relation* which a complex has to its various terms. *E.g.*, $R(-, y, z)$ is our standard way of symbolising

$$(\iota S)[R(x, y, z)\ Sx]$$

in Russell's notation. But it does not follow that we can easily find a consistent method of symbolising the *term* which has the relation R to y when this term is neither a class nor a relation.

The notation that suggests itself is $R(', y)$ for $R'y$ and $R(x, ')$ for $\breve{R}'x$. If this be adopted, R' would be represented by $R(', -)$ and $\breve{R}'$ by $R(-, ')$.

Let us now consider what would be meant by $R(', ``\beta)$. We should have $x \epsilon R(', ``\beta) . \equiv . (\exists y) . y \epsilon \beta . x = R(', y)$.

E.g., if β stands for Englishwomen and R for the relation of husband, then $R(', ``\beta)$ is the class of men who are the only husbands of Englishwomen. $R(', ``\beta)$ is thus a class which contains none of the husbands of Englishwomen who are polyandrists.

Let us now extend the notation to triadic relations.

$$\begin{array}{llllll} R(`, y, z) & \text{will be the } x & \text{such that} & R(x, y, z)! \\ R(x, `, z) & \quad ,, \quad ,, \; y & \quad ,, \quad ,, & R(x, y, z)! \\ R(x, y, `) & \quad ,, \quad ,, \; z & \quad ,, \quad ,, & R(x, y, z)!. \end{array}$$

It is easy to see how this notion can be extended by analogy with the extensions of $R(\rightarrow, y, z)$.

§ 23. *Converses of Relations.*—The notion of converses ceases to be of any great importance with our notation, for in a great many cases all that is needed of converses is expressed by the order of terms within the bracket.

Any relation will have as many 'converses' as there are permutations among its terms. Thus to any triadic relation will correspond five others. The name *converse* seems no longer applicable, it will be better to call these correlated relations. Let us start with $R(x, y, z)$, and write

$$R(x, y, z) = S(y, z, x) = T(z, x, y) = U(z, y, x) = V(y, x, z) = W(x, z, y).$$

Now in $U(z, y, x)$ the second term is in the same position as in $R(x, y, z)$, and the remaining ones are interchanged. Let us write to indicate this $R_2(z, y, x) = R(x, y, z)$. Then $U = R_2$. Similarly $V = R_3$ and $W = R_1$. It remains to symbolise S and T.

Suppose we start with the order y, z, x. Then keeping the first term fixed, and interchanging the other two, we get y, x, z, Now keep the third term fixed, and interchange the other two. We get x, y, z. We may represent S therefore as R_{31}. It is easy to see that it could equally be represented by R_{12} or R_{23}. Thus, taking R_{23}, we should first get z, y, x, and then x, y, z.

Hence $S = R_{12} = R_{23} = R_{31}$.

Now the essential point here is not that such and such numbers should be chosen, but that some pair should be chosen in direct cyclic order. Hence we might represent S by $\overrightarrow{R}$.

Similarly for T. Starting with the order z, x, y we can first keep x fixed and so get y, x, z. We can then keep z fixed and so get x, y, z. Thus $T = R_{32}$ As before we can show that

$$T = R_{13} = R_{32} = R_{21}.$$

Here the order is the inverse cyclic order. So T can be represented by $\overleftarrow{R}$. If R be triadic the five correlated relations are therefore R_1, R_2, R_3, $\overrightarrow{R}$ and $\overleftarrow{R}$. I am afraid that the notation for the relations correlated with those of higher order than the third would be very complex.