THE WIENER-KURATOWSKI PROCEDURE AND THE ANALYSIS OF ORDER

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RUSSELL once dismissed the Wiener-Kuratowski procedure for the construal of ordered pairs as classes (viz. <x, y> = def. {{x}, {x, y}}) as a trick. It is not clear what he meant,but I will argue here that there is a sense in which he was right: the use of the procedure does not eliminate the appeal to order.

The procedure enables one to introduce the pattern '<x, y>' so that (C) '<x, y> = < w, z > iff (x = w & y = z)' holds. Thus, it suffices for a treatment of ordered pairs in set-theoretical contexts. But there is an implicit claim in such a treatment—that the procedure suffices for an *analysis* of order. For, an ordered entity (an ordered pair) has purportedly been construed as an unordered entity (a class).[1] One is tempted to say that it obviously has since the *signs* '{{a}, {a, b}}' and '{{a, b}, {a}}' are signs for the same class, while '<a, b>' and '<b, a>' are signs for different ordered pairs. Kuratowski obviously did two things. First, he showed how to replace a sign that makes use of the linear ordering of constituent signs with a sign (a set sign giving a class in extension) that does not.[2] Second, he showed how to take an ordered pair as a class so as to satisfy (C). Classes are considered to be unordered even though their members may be given an order. Consider the class {a, b}. To say that it is not ordered is to say that {a, b} = {b, a}. To say that the members may be given an order is to say that some procedure may be used to establish an order among them; as trivial a procedure as enumerating them in an order. Recall Russell's well-known contrast of the class of all pairs of shoes with the class of all pairs of socks. *Being a left-shoe* could be used as a selection property. Thus, for such a class, if infinite, a "selection" axiom need not be appealed to in holding that there is a class consisting of one element from each member of the class of all pairs of shoes. Consider, next, the class of all classes of two members, such that one member is a pair-class while the other is a unit class.

[1] To give an analysis, as I take that notion here, is to show how one (supposed) kind of thing is to be taken in terms of another kind: a number as a class, a physical object as a class of sense data, a property as a class of particulars, a mental state as a physical state, a person as a collection of mental acts, and so on. In all such attempts it is understood that one cannot appeal to the kind being analysed in setting forth the analysans. Thus, one cannot take a person to be the class of all mental acts *had by the person*.

[2] The sign '{a, b}' involves a linear order, but in that '{a, b}' and '{b, a}' are signs for the same class we may say the linear ordering is not made use of.

Being a unit class is a property that can function, as Russell's *being a left-shoe* functions, to order the elements of each such pair-class. Moreover, in a perfectly clear sense it is unlike the property of *being a left-shoe* in that it may be called a logical or set-theoretical property. Where the members of a class may be ordered by employing such a property, I will speak of the class *being logically ordered.* The Wiener-Kuratowski procedure may be looked at as appealing to properties that serve to order classes logically. Is a class like {{a}, {a, b}}, then, an ordered entity? In that it is a class {{a}, {a, b}} = {{a, b}, {a}}, while, where a $\neq$ b, <a, b> $\neq$ <b, a>. Hence, we say that the class is not ordered in the sense in which <a, b> is. But, the notion of being *logically ordered*, introduced above, points to an interesting way in which a class like {{a}, {a, b}} differs from one like {a, b}. In the case of {a, b} there is no set-theoretical or logical property that may be appealed to to order the elements of the set. This difference may seem insignificant, for one may suggest that in no sense can the set {{a}, {a, b}} be taken as ordered since we can arbitrarily order the elements of the set in two ways by the use of the logical property *being a unit set.* Thus all one can do, as in the case of {a, b}, is impose an order. But, this points to the way in which sets like {{a}, {a, b}} are *used* as ordered entities. Given the signs '<a, b>' and '<b, a>', one arbitrarily takes them to stand for one ordered pair rather than the other, i.e. one recognizes the ordering of the signs to correspond to the ordering of the elements. Likewise, one arbitrarily construes the ordered pair <a, b> in terms of a set like {{a}, {a, b}}, or a variety of other alternatives. Finally, if, following Kuratowski, one uses sets like {{a}, {a, b}} and {{b}, {a, b}}, one *chooses* which of {{a}, {a, b}} or {{b}, {a, b}} is to represent <a, b> and which will represent <b, a>. This shows that in the construal of <a, b> as {{a}, {a, b}} one implicitly takes a and b in an ordering, for one takes the element in the unit set as the *first* element. We can see that in a simple way. Consider '$\{x_1, x_2, \ldots, x_n\}$' to be a function sign standing for a function, f_1, that yields a class as value for arguments $x_1, x_2, \ldots, x_n$—the class whose members are the arguments. One may say that the function f_1 is not an *ordering function* in two senses. (1) A class, whose members are the arguments, is the value; hence any sign of the form '$\{x_\alpha, x_\beta, \ldots, x_\nu\}$', which is the result of a permutation of the signs for the elements of the class (the arguments) is such that $\{x_\alpha, x_\beta, \ldots, x_\nu\} = \{x_1, x_2, \ldots, x_n\}$. (2) Taking the arguments in any order to be operated on by the function yields the same value. Consider, next, a two-term function, f_2, represented by '$\{\{x_1\}, \{x_1, x_2\}\}$'. Take a and b as arguments. f_2 also yields a class as value. The function is not an ordering function and the value is not ordered in the sense that a class is the value, and, hence, {{a}, {a, b}} = {{a, b}, {a}}. But f_2 is an ordering function and the class may be considered ordered in that the same class does not result as the value when the arguments are taken in a different order. The class, {{a}, {a, b}}, is the value of an ordering function for arguments a and b, though it is not the value of an ordering function for arguments {a} and {a, b}, in the sense in which I am using the notion of an *ordering function*.

It will not do to suggest that we consider {{a}, {a, b}} as the "result" of two applications of the class function f_1 so that the first application yields the classes {a} and {a, b} and another yields the class {{a}, {a, b}}. For, the first application does not employ the function f_1, since, given a and b as arguments, we must take them in an order to get {a} and {a, b}, rather than {b} and {a, b}. Thus, if we consider $<x_1, x_2>$ to be a function that yields an ordered pair for arguments a and b, such a function is like f_2, and not like f_1, in that it is an ordering function in the above sense. The point can be made more emphatic if we take f_2 to be a three-term function, which yields {{a}, {a, b}} when the sequence (a, a, b) is taken as argument, and {{b}, {a, b}}, with (b, a, b) as argument. We may then note that we have sequences that differ not only in the arrangement of terms but in the repetition of *different* terms. These *two relevant differences* point to the implicit appeal to order in the *use* of a class like {{a}, {a, b}}.

It will also not do to argue that the point suggested here is vacuous, in that no function which will not be an ordering function in the above sense could be used to yield a set that would suffice for the construal of an ordered pair. The objection merely serves to emphasize a clear sense in which order has been presupposed and not analysed by Kuratowski's procedure. If the objection is pushed to suggest that on my argument no analysis of relational order could be presented, I think the point made is correct, but neither a *reductio* nor an objection.

To object that the Wiener-Kuratowski procedure has nothing to do with a purported *analysis* of order, but merely with *modelling* relational statements, avoids the problem. That one can provide a model of a certain kind is not at issue; just as it is not generally at issue that Russell provided a model of the Dedekind-Peano postulates. To provide a model is not to furnish an analysis.[3] Likewise, that alternative models exist does not imply that one, or more than one, model cannot be used as a basis for an analysis. The point here is not that there are alternative models of the Wiener-Kuratowski type, but that in constructing a model along such lines one implicitly appeals to an ordering of the elements a and b in the use of sets like {{a}, {a, b}}.

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[3] On this point see my 'Peano, Russell, and Logicism,' ANALYSIS, 16.5, April 1956, pp. 118–20.