Weaves

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In this paper we introduce the notion of a weave. We will show how to combine weaves in various ways to get other weaves. One such method of combination will give us a structure that generalizes the idea of an infinite win-lose game with perfect information (henceforth referred to as a Gale-Stewart game), but still has many of the properties that Gale-Stewart games have. (For instance, the determinateness properties of Gale-Stewart games can be translated into the language of weaves.) All terms used in this paper in connection with Gale-Stewart games are defined in [1] and [2].

We begin with a definition:

Let $\mathfrak{L}$ and $\mathcal{R}$ be subsets of P(D), $(\mathfrak{L}, \mathcal{R})$ is called a *wave* of D if the following conditions are satisfied

1. $\forall A \in \mathfrak{L} \quad \forall B \in \mathcal{R} \quad \overline{\overline{A \cap B}} = 1$

2. $\forall d \in D \quad \exists A \in \mathfrak{L} \quad \exists B \in \mathcal{R} \quad \{d\} = A \cap B$

A weave will be compared (in a sense to be explained later) to a single move of a Gale-Stewart game.

We proceed to formulate a notion of "union" of weaves and two motions of "product" of weaves.

Proposition 1. Let $(\mathfrak{L}, \mathcal{R})$ be a weave of D and $\{D_d | d \in D\}$ be a family of mutally disjoint sets. Let $(\mathfrak{L}_d, \mathcal{R}_d)$ be a weave of D_d for every $d \in$ D. Define

$$\mathfrak{L} = \{ \bigcup_{d \in A} f(d) | A \in \mathfrak{L} \wedge f \in \prod_{d \in D} \mathfrak{L}_d \}$$

$$\mathcal{R} = \{ \bigcup_{d \in B} g(d) | B \in \mathcal{R} \wedge g \in \prod_{d \in D} \mathcal{R}_d \}$$

Then $(\mathfrak{L}, \mathcal{R})$ is a weave of $\tilde{D} = \bigcup_{d \in D} D_d$. $(\mathfrak{L}, \mathcal{R})$ is called a *sum* of $\{(\mathfrak{L}_d, \mathcal{R}_d) | d \in D\}$.

Proof. Let $A \in \mathfrak{L}$, $B \in \mathcal{R}$, $f \in \prod_{d \in D} \mathfrak{L}_d$ and $g \in \prod_{d \in D} \mathcal{R}_d$.

$$(\bigcup_{d \in A} f(d)) \cap (\bigcup_{d \in B} g(d)) = f(d_0) \cap g(d_0)$$

where $\{d_0\} = A \cap B$. Moreover

$$\overline{\overline{f(d_0) \cap g(d_0)}} = 1 .$$

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Let $p \in \bigcup_{d \ni D} D_d$. There exists a unique $d_0 \in D$ s.t. $p \in D_{d_0}$. Let $A \in \mathfrak{L}$ & $B \in \mathcal{R}$ satisfy $A \cap B = \{d_0\}$. Then there exists $A^\circ \in \mathfrak{L}_{d_0}$, $B^\circ \in \mathcal{R}_{d_0}$ s.t. $A^\circ \cap B^\circ = \{p\}$. Take f & g to satisfy $f(d_0)=A^\circ$ & $g(d_0) = B^\circ$.

Proposition 2. Let $(\mathfrak{L}_i, \mathcal{R}_i)$ be a weave of D_i for every $i \in I$. Define

$$D = \prod_{i \in I} D_i\,,$$

$$A_f = \prod_{i \in I} f(i) \qquad \text{for} \qquad f \in \prod_{i \in I} \mathfrak{L}_i$$

and

$$B_g = \prod_{i \in I} g(i) \qquad \text{for} \qquad g \in \prod_{i \in I} \mathcal{R}_i\,.$$

Finally define

$$\mathfrak{L} = \{A_f \mid f \in \prod_{i \in I} \mathfrak{L}_i\} \qquad \text{and} \qquad \mathcal{R} = \{B_g \mid g \in \prod_{i \in I} \mathcal{R}_i\}\,.$$

Then $(\mathfrak{L}, \mathcal{R})$ is a weave of D. This weave is called a *product* of $\{(\mathfrak{L}_i, \mathcal{R}_i) \mid i \in I\}$ and denoted by $\prod_{i \in I} (\mathfrak{L}_i, \mathcal{R}_i)$.

Proof. Let $f \in \prod_{i \in I} \mathfrak{L}_i$ and $g \in \prod_{i \in I} \mathcal{R}_i$, $A_f \cap B_g = \prod_{i \in I} (f(i) \cap g(i))$. Since $\overline{\overline{f(i) \cap g(i)}} =1$ for every $i \in I$, $\overline{\overline{A_f \cap B_g}} = 1$. Let $h \in D$. For every $i \in I$, there exists $A_i \in \mathfrak{L}_i$ and $B_i \in \mathcal{R}_i$ s.t. $\{h(i)\} = A_i \cap B_i$. Define f by $f(i)=A_i$ and g by $g(i)=B_i$. Then $A_f \cap B_g = \{h\}$.

Remark. Let $(\mathfrak{L}, \mathcal{R})$ be a weave of D. If $A_1 \subseteq A_2$ and $A_1, A_2 \in \mathfrak{L}$, then $A_1 = A_2$.

Proof. Suppose $p \in A_2 - A_1$. Then let $B \in \mathcal{R}$ satisfy $p \in B$. Then

$$\overline{\overline{A_1 \cap B}} = \overline{\overline{A_2 \cap B}} = 1$$

is a contradiction.
In the same way, if $B_1 \subseteq B_2$ and $B_1, B_2 \in \mathcal{R}$, then

$$B_1 = B_2\,.$$

Definition. Let $(\mathfrak{L}, \mathcal{R})$ be a weave of D. $(\mathfrak{L}, \mathcal{R})$ is called *normal* if for every subset X of D there exist $A \in \mathfrak{L}$ and $B \in \mathcal{R}$ s.t.

$$\text{either} \qquad A \subseteq X \qquad \text{or} \qquad B \subseteq D - X\,.$$

Let $\mathfrak{F} \subseteq P(D)$. Then $(\mathfrak{L}, \mathcal{R})$ is called $\mathfrak{F}$-*normal* if for every $X \in \mathfrak{F}$ there exists $A \in \mathfrak{L}$ and $B \in \mathcal{R}$ s.t.

$$\text{either} \qquad A \subseteq X \qquad \text{or} \qquad B \subseteq D - X\,.$$

Definition. Let $\mathfrak{F}_1 \subseteq P(D_1)$ and $\mathfrak{F}_2 \subseteq P(D_2)$. A *tensor product of* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ which is denoted by $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ is defined by the following.

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \{\langle d_1, d_2 \rangle \mid d_1 \in F_1 \wedge d_2 \in f(d_1) \wedge F_1 \in \mathfrak{F}_1 \wedge (f : F_1 \to \mathfrak{F}_2)\}\,.$$

Proposition. $(\mathfrak{F}_1 \otimes \mathfrak{F}_2) \otimes \mathfrak{F}_3 \cong \mathfrak{F}_1 \otimes (\mathfrak{F}_2 \otimes \mathfrak{F}_3)$

Proposition. $P(D_1) \otimes P(D_2) = P(D_1 \times D_2)$

Rmeark: The element of $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ which is obtained from $F_1 \in \mathfrak{F}_1$ & $f: F_1 \to \mathfrak{F}_2$ is denoted by F_1^f

Proposition 3. Let $(\mathfrak{L}_i, \mathcal{R}_i)$ be a weave of D_i for $i=1, 2$. Define

$$\mathfrak{L} = \mathfrak{L}_1 \otimes \mathfrak{L}_2 \qquad \text{and} \qquad \mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 .$$

Then $(\mathfrak{L}, \mathcal{R})$ is a wave of $D=D_1 \times D_2$. $(\mathfrak{L}, \mathcal{R})$ is called a *tensor product of* $(\mathfrak{L}_1, \mathcal{R}_1)$ *and* $(\mathfrak{L}_2, \mathcal{R}_2)$.

Proof. Since $\mathfrak{L}_1 \times \mathfrak{L}_2 \subseteq \mathfrak{L}$ and $\mathcal{R}_1 \times \mathcal{R}_2 \subseteq \mathcal{R}$, it suffices to show that for every $A \in \mathfrak{L}$ and $B \in \mathcal{R}$

$$\overline{\overline{(A \cap B)}} = 1 .$$

Let A be obtained from $A_1 \in \mathfrak{L}_1$ and $f_1: A_1 \to \mathfrak{L}_2$ and B be obtained from $B_1 \in \mathcal{R}_1$ and $g_1: B_1 \to \mathcal{R}_2$.

Let $A_1 \cap B_1 = \{d_1\}$ and $f_1(d_1) = A_2$ and $g_1(d_1) = B_2$. Then the proposition is obvious since $A \cap B = \{d_i\} \times (A_2 \cap B_2)$.

Definition. Let D be a non-empty set. A weave $(\mathfrak{L}, \mathcal{R})$ of D is called *trivial* if $(\mathfrak{L}, \mathcal{R})$ is one of the following:

a) $\mathfrak{L} = \{\{d\} \mid d \in D\}$ and $\mathcal{R} = \{D\}$

b) $\mathfrak{L} = \{D\}$ and $\mathcal{R} = \{\{d\} \mid d \in D\}$.

A trivial weave of D is normal.

Now we answer some questions of the following type: Which methods of combining weaves preserve the property of being normal ? Which preserve the property of being $\mathfrak{F}$-normal, for some particular kinds of families $\mathfrak{F}$?. In the case of tensor products, this turns out to be equivalent to questions of determinateness of Gale-Stewart games.

Proposition 4. Let $(\mathfrak{L}_i, \mathcal{R}_i)$ be a weave of D_i for $i=1, 2$, and $(\mathfrak{L}, \mathcal{R})$ be the product of $(\mathfrak{L}_1, \mathcal{R}_1)$ and $(\mathfrak{L}_2, \mathcal{R}_2)$. If $(\mathfrak{L}_i, \mathcal{R}_i)$ is $\mathfrak{F}_i$-normal for $i=1, 2$, then $(\mathfrak{L}, \mathcal{R})$ is $\mathfrak{F}$-normal, where $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2$.

Proof. Let $X_1 \subseteq D_1$ and $X_2 \subseteq D_2$. Let $A_1 \in \mathfrak{L}_1$, $B_1 \in \mathcal{R}_1$, $A_2 \in \mathfrak{L}_2$, $B_2 \in \mathcal{R}_2$ satisfy the following conditions.

1) either $A_1 \subseteq X_1$ or $B_1 \subseteq D_1 - X_1$

2) either $A_2 \subseteq X_2$ or $B_2 \subseteq D_2 - X_2$

Case 1) $A_1 \subseteq X_1$ & $A_2 \subseteq X_2$.

Then $A_1 \times A_2 \subseteq X_1 \times X_2$.

Case 2) $B_1 \subseteq D_1 - X_1$.

$$B_1 \times B_2 \subseteq (D_1 - X_1) \times B_2 \subseteq (D_1 - X_1) \times D_2$$
$$= D_1 \times D_2 - X_1 \times D_2 \subseteq D_1 \times D_2 - X_1 \times X_2$$

Case 3) $B_2 \subseteq D_2 - X_2$ (similar).

Proposition 5. Let $(\mathfrak{L}_i, \mathcal{R}_i)$ be a weave of D_i for $i=1, 2$, and $(\mathfrak{L}, \mathcal{R})$ be

$$(\mathfrak{L}_1, \mathcal{R}_1) \otimes (\mathfrak{L}_2, \mathcal{R}_2).$$

If $(\mathfrak{L}_i, \mathcal{R}_i)$ is $\mathfrak{F}_i$-normal for $i=1, 2$, and $\mathfrak{F}_1 = P(D_1)$ then $(\mathfrak{L}, \mathcal{R})$ is $\mathfrak{F}$-normal here $\mathfrak{F} = \mathfrak{F}_1 \otimes \mathfrak{F}_2$.

Proof. Let $X \in \mathfrak{F}$ and X be obtained from $X_1 \in \mathfrak{F}_1$ and $f: X_1 \to \mathfrak{F}_2$. There exists $A_1 \in \mathfrak{L}_1$ and $B_1 \in \mathcal{R}_1$ such that either $A_1 \subseteq X_1$ or $B_1 \subseteq D_1 - X_1$

Case 1) $B_1 \subseteq D_1 - X_1$.

Take any $B_2 \in \mathcal{R}_2$

$$B_1 \times B_2 \subseteq (D_1 - X_1) \times D_2 = D_1 \times D_2 - X_1 \times D_2 \subseteq D_1 \times D_2 - X$$

Case 2) $A_1 \subseteq X_1$.

For $d_1 \in X_1$ there exist $A_{d_1} \in \mathfrak{L}_2$ and $B_{d_1} \in \mathcal{R}_2$ such that either $A_{d_1} \subseteq f(d_1)$ or $B_{d_1} \subseteq D_2 - f(d_1)$. Let $Y = \{d_1 \in X_1 \mid A_{d_1} \subseteq f(d_1)\}$. If there exists $A \in \mathfrak{L}_1$ such that $A \subseteq Y$, then A and $\{d \to A_d \mid d \in A)$ make an element of $\mathfrak{L}$ included in X. Suppose there exists $B \in \mathcal{R}_1$ such that

$$B \subseteq D_1 - Y$$

We define $g: B \to \mathcal{R}_2$ as follows:

$$g(d_1) \text{ is any member of } \mathcal{R}_2 \text{ if } d_1 \notin X_1$$
$$g(d_1) \text{ is } B_{d_1} \text{ if } d_1 \in X_1.$$

Suppose $\langle d_1, d_2 \rangle \in B^g \cap X$. Then

$$d_2 \in g(d_1) \subseteq D_2 - f(d_1) \wedge d_2 \in f(d_1)$$

which is a contradiction.

Definition. Let $\mathfrak{F} \subseteq P(D)$ and $\mathfrak{F}_d \subseteq P(D_d)$ for every $d \in D$. We define

$$\mathfrak{F} - \bigcup_{d \in D} \mathfrak{F}_d = \{\bigcup_{d \in A} f(d) \mid A \in \mathfrak{F} \wedge f \in \prod_{d \in D} \mathfrak{F}_d\}.$$

In the definition of the sum of weaves,

$$\tilde{\mathfrak{L}} = \mathfrak{L} - \bigcup_{d \in D} \mathfrak{L}_d \qquad \text{and} \qquad \tilde{\mathcal{R}} = \mathcal{R} - \bigcup_{d \in D} \mathcal{R}_d.$$

Proposition 6. Let $\{D_d \mid d \in D\}$ be mutually disjoint, $(\mathfrak{L}_d, \mathcal{R}_d)$ be an $\mathfrak{F}_d$-normal weave of D_d and $(\mathfrak{L}, \mathcal{R})$ be an $\mathfrak{F}$-normal weave of D, where $\mathfrak{F} = P(D)$. Define $(\tilde{\mathfrak{L}}, \tilde{\mathcal{R}})$ to be the sum of $(\mathfrak{L}_d, \mathcal{R}_d)$. Then $(\tilde{\mathfrak{L}}, \tilde{\mathcal{R}})$ is an $\tilde{\mathfrak{F}}$-normal weave of $\tilde{D}$, where

$\tilde{\mathfrak{F}} = \mathfrak{F} - \bigcup_{d \in D} \mathfrak{L}_d$ and $\tilde{D} = \bigcup_{d \in E} D_d$

Proof. By proposition 1, it suffices to show the $\tilde{\mathfrak{F}}$-normality of $(\tilde{\mathfrak{L}}, \tilde{\mathcal{R}})$.

Let $X \in \tilde{\mathfrak{F}}$. Then X is of the form

$$\bigcup_{d \in D} X_d$$

where $X_d \in \mathfrak{F}_d$. By the $\mathfrak{F}_d$-normality of $(\mathfrak{L}_d, \mathcal{R}_d)$ we have

$$\forall d \in D \ \exists A_d \in \mathfrak{L}_d \ \exists B_d \in \mathcal{R}_d \ \text{ s.t. } \ A_d \subseteq X_d \ \text{ or } \ B_d \subseteq D_d - X_d .$$

Let $Y = \{d \in D \mid A_d \subseteq X_d\}$. Then

$$\exists A \in \mathfrak{L} \ \exists B \in \mathcal{R} \ \text{ s.t. } \ A \subseteq Y \ \text{ or } \ B \subseteq D - Y .$$

Case 1) $A \subseteq Y$.

Let $\tilde{A} = \bigcup_{d \in A} A_d$. Then $\tilde{A} \in \tilde{\mathfrak{L}}$ by definition of the sum and $\tilde{A} \subseteq X$.

Case 2) $B \subseteq D - Y$.

For $d \in B$ we have $B_d \subseteq D_d - X_d$. Define $\tilde{B} = \bigcup B_d$. Then $\tilde{B} \in \tilde{\mathcal{R}}$. Also

$$\bigcup_{d \in B} B_d \subseteq \bigcup_{d \in B} (D_d - X_d) = \bigcup_{d \in B} (D_d - X) \subseteq D - X .$$

The added condition that $\mathfrak{F} = P(D)$ is necessary for the proof of proposition 6. If we let

$$\mathfrak{L} = \{\{a, b\}, \ \{c, d\}, \ \{a, c\}\}$$

$$\mathcal{R} = \{\{a, d\}, \ \{b, c\}\}$$

$$\mathfrak{L}_a = \{\{0_a, 1_a\}\} \qquad \mathfrak{L}_b = \{\{0_b\}, \ \{1_b\}\}$$

$$\mathcal{R}_a = \{\{0_a\}, \ \{1_a\}\} \qquad \mathcal{R}_b = \{\{0_b, 1_b\}\}$$

$$\mathfrak{L}_c = \{\{0_c, 1_c\}\} \qquad \mathfrak{L}_d = \{\{0_d\}, \ \{1_d\}\}$$

$$\mathcal{R}_c = \{\{0_c\}, \ \{1_c\}\} \qquad \mathcal{R}_d = \{\{0_d, 1_d\}\}$$

Then $\mathfrak{F} \neq P(D)$ and the sum $(\tilde{\mathfrak{L}}, \tilde{\mathcal{R}})$ is not $\tilde{\mathfrak{F}}$-normal.

Remark: If $\{D_d \mid d \in D\}$ is mutually disjoint and $\tilde{D} = \bigcup_{d \in D} D_d$, then $P(\tilde{D}) = P(D) - \bigcup_{d \in D} P(D_d)$.

To what extent does the infinite tensor product preserve the property of being normal ? To answer this we convert from the language of weaves to the language of games.

Let $(\mathfrak{L}_i, \mathcal{R}_i)$ be a normal weave of D_i for $i < \omega$. Define a game G as follows: Let X be a subset of $\prod_i D_i$.

Stage 0: Players I and II *simultaneously* choose sets $A_0 \in \mathfrak{L}_0$ and $B_0 \in \mathcal{R}_0$ respectively.

⋮

Stage i: Players I and II *simultaneously* choose sets $A_i \in \mathfrak{L}_i$ and $B_i \in \mathcal{R}_i$ respectively.

⋮

For each i, let $A_i \cap B_i = \{d_i\}$. We say that player I wins the game if and only if the sequence $\langle d_0, d_1, d_2, \cdots \rangle$ is in X. Otherwise, player II wins. This is called a *weave game with underlying set X*.

(Remark: This weave game is a generalization of the Gale-Stewart game. A Gale-Stewart game can be thought of as a weave game using only the trivial weave $\mathfrak{L}_i = \{\{d\} \mid d \in D_i\}$, $\mathcal{R}_i = \{D_i\}$ for stages, i, where palyer I is expected to choose, and $\mathfrak{L}_j = \{D_j\}$, $\mathcal{R}_j = \{\{d\} \mid d \in D_j\}$ for stages, j, where player II is expected to choose.)

Let $\tilde{\mathfrak{L}} = \mathfrak{L}_0 \otimes \mathfrak{L}_1 \otimes \mathfrak{L}_2 \otimes \cdots$, $\tilde{\mathcal{R}} = \mathcal{R}_0 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \cdots$, and $\bar{D} = D_0 \times D_1 \times D_2 \times \cdots$.

Now we translate "tensor product" into game terminology: A *play* of the game G is an element of $\bar{D}$. A *finite play* of the game G is an initial segment of some play. A *strategy* for player I (II) in the game G is a function σ (whose domain is the set of finite plays), such that for all $\langle d_0, \cdots, d_n \rangle$, $\sigma(\langle d_0, \cdots, d_n \rangle) \in \mathfrak{L}_{n+1}(\mathcal{R}_{n+1})$. (We will use these terms in connection with Gale-Stewart games also; see [1] for definitions.) Given a strategy σ, let $\hat{\sigma}$ be the set of all plays of G that can result from player I's using the strategy σ throughout the game.

Notice that $\tilde{\mathfrak{L}} = \{\hat{\sigma} \mid \sigma$ a strategy for I in $G\}$. This is because every set $\tilde{A}$ in $\tilde{\mathfrak{L}}$ is of the form $A_0^{\langle f_1, f_2, \cdots \rangle}$ where $A_0 \in \mathfrak{L}_0$, $f_1 \colon A_0 \to \mathfrak{L}_1$, $f_2 \colon A_0^{f_1} \to \mathfrak{L}_2$, etc. So $\tilde{A} = \hat{\sigma}$, where σ is a strategy satisfying

$$\sigma(\varLambda) = A_0$$

$$\sigma(\langle d_0 \rangle) = f_1(d_0) \qquad \forall d_0 \in A_0$$

$$\vdots$$

$$\sigma(\langle d_0, \cdots, d_i \rangle) = f_{i+1}(\langle d_0, \cdots, d_i \rangle) \qquad \forall \langle d_0, \cdots, d_i \rangle \in A_0^{\langle f_1, \cdots, f_i \rangle}$$

Conversely, given any strategy σ for I, Define $A_0, f_1, \cdots, f_{i+1}, \cdots$ according to the above equations. Then $\hat{\sigma} = A_0^{\langle f_1, f_2, \cdots \rangle}$.

Similarly, $\tilde{\mathcal{R}} = \{\hat{\tau} \mid \tau$ a strategy for II in $G\}$. (This is why we chose to make G a simultaneous-play game rather than a Gale-Stewart game. If we'd made G a Gale-Stewart game, this last claim would not have been true.)

We say that a game G with underlying set X is *determined* if and only if one of the players has a winning strategy. This is equivalent to

$$\exists \sigma \text{ (strategy for I)} \quad \exists \tau \text{ (strategy for II)} \quad \hat{\sigma} \subseteq X \vee \hat{\tau} \subseteq \bar{D} - X$$

This in turn is equivalent to

$$\exists \tilde{A} \in \tilde{\mathfrak{L}}\ \exists \tilde{B} \in \tilde{\mathcal{R}} \qquad \tilde{A} \subseteq X \vee \tilde{B} \subseteq \tilde{D} - X$$

Therefore, $(\tilde{\mathfrak{L}}, \tilde{\mathcal{R}})$ is $\mathfrak{F}$-normal, for some particular family of sets $\mathfrak{F}$, if and only if for every X in $\mathfrak{F}$, the game on $(\tilde{\mathfrak{L}}, \tilde{\mathcal{R}})$ with underlying set X is determined.

Next we reduce the problem further to that of determinateness of Gale-Stewart games. We consider a weave game G and create from it an "equivalent" Gale-Stewart game G^*.

The game G^*: at stage i, player I chooses a set A_i in $\mathfrak{L}_i$, *and then* player II chooses a set B_i in $\mathcal{R}_i$.

Again, an element $\langle d_0, d_1, \cdots \rangle$ of $\tilde{D}$ is formed and player I wins if and only if $\langle d_0, d_1, \cdots \rangle \in X$.

Lemma 1. If player I has a winning strategy in G^*, then he has a winning strategy in G.

Proof. Player I should play G using essentially the same strategy that he uses to win G^*.

Let σ^* be a winning strategy for I in G^*.

Let $\langle d_0, \cdots d_n \rangle$ be a finite play of G. Choose $A_0, B_0, A_1, B_1, \cdots, A_n, B_n$ such that $A_i \cap B_i = \{d_i\}$ for all $i=1, \cdots, n$. Define $\sigma(\langle d_0, \cdots, d_n \rangle) = \sigma^*(\langle A_0, B_0, \cdots, A_n, B_n \rangle)$. (Notice that a "finite play" of G^* is an initial segment of an element in $\mathfrak{L}_0 \times \mathcal{R}_0 \times \mathfrak{L}_1 \times \mathcal{R}_1 \times \cdots$. This is the only "difficulty" in applying a strategy for I in G^* to the game G.)

Then σ is a winning strategy for I in G.

Lemma 2. If player II has a winning strategy in G^*, then he has a winning strategy in G.

Proof. (Intuitive Idea): Games G and G^* are essentially the same except that player II seems to have an extra advantage in G^* that he doesn't have in G – that of knowing, at stage i, what player I's move for stage i will be before having to choose his own move for stage i. This turns out not to be an advantage at all (because of the normality of $(\mathfrak{L}_i, \mathcal{R}_i)$). For instance, at stage O, we let D_0^{II} be the set of all d in D_0 that players I and II can form (jointly) as part of τ^*- a winning strategy for II in G^*. It turns out that $\exists B \in \mathcal{R}_0$ s.t. $B \subseteq D_0^{II}$. In playing the game G, player II can simply choose B at stage O, without knowing what player I's choice for stage O will be.

(Details:) Let f be the function that maps the plays of G^* onto the plays of G in the expected way (i.e. $f(\langle A_0, B_0, A_1, B_1, \cdots \rangle) = \langle d_0, d_1, \cdots \rangle$ where $A_i \cap B_i = \{d_i\}$ for all $i < \omega$.) Let τ^* be a winning strategy for II in G^*.

$\hat{\tau}^*$ is a II-imposed subgame (see [2]) of G^* in which all plays are wins for player II. Therefore, $f(\hat{\tau}^*)$ is a "subgame" of G in which all plays are wins for player II. We will define a strategy τ for II in G in such a way that $\hat{\tau} \subseteq f(\hat{\tau}^*)$. (Note that since $f(\hat{\tau}^*)$ is a closed set, we only have to define τ so as to make any *finite play* of $\mathfrak{r}$ extendable to an element of $f(\mathfrak{r}^*)$.) Thus τ will be a winning strategy for II in G.

Let $D_0{}^{II}$ be the set of *Oth* entries of elements of

$$f(\mathfrak{r}^*). \quad [\text{i.e. } D_0{}^{II}=\{d_0|(\exists d_1, d_2, \cdots)[\langle d_0, d_1, d_2, \cdots\rangle \in f(\mathfrak{r}^*)]\}]$$

Claim: $\exists B \in \mathcal{R}_0\ B \subseteq D_0{}^{II}$.

If not then by normality $\exists A \in \mathfrak{L}_0\ A \subseteq D_0 - D_0{}^{II}$. This would imply that there is a move of player I in G^* (that of choosing the set A) that forces the game out of $\mathfrak{r}^*$, contradicting the fact that $\mathfrak{r}^*$ is a II-imposed subgame of G^*.

So define $\tau(A)=B$.

Now assume that G is in its *ith* stage and $\langle d_0, \cdots, d_{i-1}\rangle$ has already been played and (hypothesis of induction) that $\langle d, \cdots, d_{i-1}\rangle$ is an initial segment of some element of $f(\mathfrak{r}^*)$.

Let $D_i{}^{II}\langle d_0, \cdots, d_{i-1}\rangle$ be the set of *ith* entries of those elements of $f(\mathfrak{r}^*)$ that have $\langle d,_0 \cdots, d_{i-1}\rangle$ as an initial segment. [i.e. $D_i{}^{II}\langle d_0, \cdots, d_{i-1}\rangle=\{d_i|(\exists d_{i+1}, d_{i+2}, \cdots)[\langle d_0, \cdots, d_{i-1}, d_i, d_{i+1}, d_{i+2}, \cdots\rangle \in f(\mathfrak{r}^*)]\}]$. As before, $\exists B \in \mathcal{R}_i\ B \subseteq D_i{}^{II}\langle d_0, \cdots, d_{i-1}\rangle$. Define $\tau(\langle d_0, \cdots, d_{i-1}\rangle)$ to be this B.

$\mathfrak{r} \subseteq f(\mathfrak{r}^*)$, so τ is a winning strategy for II in G. This completes the proof..

Collecting the results of Lemmas 1 and 2, we have the following:

Lemma 3: If G^* is determined, so is G.

It is shown in [3] and [4] that any Gale-Stewart game G^*, whose underlying set X is a Borel set (as a subset of $\tilde{D}$), is determined. We therefore have, by Lemma 3, that any weave game, whose underlying set is a Borel set, is determined. Translating this back into the language of tensor products, we have:

Proposition 7. Let $(\mathfrak{L}_i, \mathcal{R}_i)$ be a normal weave of D_i for $i<\omega$. Let

$$\tilde{\mathfrak{L}}=\mathfrak{L}_0\otimes\mathfrak{L}_1\otimes\mathfrak{L}_2\otimes\cdots$$

$$\tilde{\mathcal{R}}=\mathcal{R}_0\otimes\mathcal{R}_1\otimes\mathcal{R}_2\otimes\cdots$$

and

$$\tilde{D}=D_0\times D_1\times D_2\times\cdots.$$

Let X be a Boreal subset of $\tilde{D}$. Then there exixt $\tilde{A} \in \tilde{\mathfrak{L}}$ and $\tilde{B} \in \tilde{\mathcal{R}}$ such that either $\tilde{A} \subseteq X$ or $\tilde{B} \subseteq \tilde{D}-X$.

There is a similar theorem for the sum of weaves, but to do this we have to rewrite the definition of sum so that we can define the infinitely iterated sum.

Let $(\mathfrak{L}, \mathcal{R})$ be a weave of D, $\{D_d|d \in D\}$ be a family of mutually disjoint sets, and let $(\mathfrak{L}_d, \mathcal{R}_d)$ be a weave of D_d for each d in D.

For each A in $\mathfrak{L}$ and each $f \in \prod_{d\in D} \mathfrak{L}_d$ let $A^f=\{e|(\exists d \in A)[e \in f(d)]\}$. The sum of $\{(\mathfrak{L}_d, \mathcal{R}_d)|d \in D\}$ is $\tilde{\mathfrak{L}}=\{A^f|A \in \mathfrak{L} \wedge f \in \prod_{d\in D}\mathfrak{L}_d\}$.

Now define ${}^fA$ to be $\{\langle d, e\rangle|d \in A \wedge e \in f(d)\}$. Let the *sum'* of $\{(\mathfrak{L}_d, \mathcal{R}_d)|d \in D\}$

be $\tilde{\mathfrak{L}}'=\{{}^{f}A \mid A\in\mathfrak{L}\wedge f\in\prod_{d\in D}\mathfrak{L}_d\}$.

$\tilde{\mathfrak{L}}$ and $\tilde{\mathfrak{L}}'$ are isomorphic, by the correspondance

$$e\longleftrightarrow\langle d, e\rangle \text{ where } e\in D_d .$$

The advantage of working with $\tilde{\mathfrak{L}}'$ instead of $\tilde{\mathfrak{L}}$ is that the process of sum′-ing can be iterated infinitely. Let $(\mathfrak{L}, \mathcal{R})$ be a weave of D_0,

$\{D_{d_0} \mid d_0\in D_0\}$ a collection of mutually disjoint sets,

$(\mathfrak{L}_{d_0}, \mathcal{R}_{d_0})$ a weave of D_{d_0} for each $d_0\in D_0$,

$D_1=\bigcup_{d_0\in D_0} D_{d_0}$,

⋮

$\{D_{d_i} \mid d_i\in D_i\}$ a collection of mutually disjoint sets,

$(\mathfrak{L}_{d_i}, \mathcal{R}_{d_i})$ a weave of D_{d_i} for each $d_i\in D_i$,

$D_{i+1}=\bigcup_{d_i\in D_i} D_{d_i}$

⋮

We define the iterated sum of all these weaves

For each $A\in\mathfrak{L}$, $f_0\in\prod_{d_0\in D_0}\mathfrak{L}_{d_0},\cdots,$

$f_n\in\prod_{d_n\in D_n}\mathfrak{L}_{d_n},\cdots,$ define $\langle f_0,\cdots,f_n,\cdots\rangle_A$

to be

$$\{\langle a_0, a_1,\cdots\rangle \mid a_0\in A\wedge a_1\in f_1(a)\wedge a_2\in f_2(a_1)\wedge\cdots\}$$

Let $\tilde{\mathfrak{L}}$ be $\{{}^{\langle f_0,\cdots,f_n,\cdots\rangle}A \mid A\in\mathfrak{L}\wedge(\forall i)\,[f_i\in\prod_{d_i\in D_i}\mathfrak{L}_{d_i}]\}$.

In a similar way define $\tilde{\mathcal{R}}$ to be

$$\{{}^{\langle g_0,\cdots,g_n,\cdots\rangle}B \mid B\in\mathcal{R}\wedge(\forall i)\,[g_i\in\prod_{d_i\in D_i}\mathcal{R}_{d_i}]\} .$$

Proposition 8. $(\tilde{\mathfrak{L}}, \tilde{\mathcal{R}})$ is a weaves.

Once again we define a game G:

P is a subset of $\prod_i D_i$. $\langle a_0, a_1,\cdots\rangle$ is in P if and only if $a_0\in D_0$, $a_1\in D_{a_0}$, $a_2\in D_{a_1}$, etc. P is the set of plays of the game G.

Let X be a subset of P.

Stage 0: Players I and II *simultaneously* choose sets $A\in\mathfrak{L}$ and $B\in\mathcal{R}$ respectively. Let $A\cap B=\{d_0\}$.

⋮

Stage $i+1$: Players I and II *simultaneoulsy* choose sets $A_{d_i}\in\mathfrak{L}_{d_i}$ and $B_{d_i}\in\mathcal{R}_{d_i}$ respectively. Let $A_{d_i}\cap B_{d_i}=\{d_{i+1}\}$.

⋮

Other definitions for G follow in a way similar to that for tensor product.

As in the case of tensor products we have that

$\tilde{\mathfrak{L}}=\{\hat{\sigma}|\sigma$ a strategy for I in $G\}$ and $\tilde{\mathcal{R}}=\{\hat{\tau}|\tau$ a strategy for II in $G\}$ and so lemmas analogous to lemmas 1 and 2 for tensor products, and proposition 7, can be proved for iterated sum. The slight advantage of the iterated sum over the tensor product is that in the iterated sum game, at any stage in the game, the choices available to players I and II can depend on the finite play of the game up to that stage.

Proposition 9. Let $(\mathfrak{L}_i, \mathcal{R}_i)$ be a normal weave of D_i for $i=1, 2$. Define $\mathfrak{L}=\mathfrak{L}_1\otimes\mathfrak{L}_2$, $\mathcal{R}=\mathcal{R}_1\otimes\mathcal{R}_2$ and $D=D_1\times D_2$. Let $X\colon D_2\to P(D_1)$ i.e. for every $d_2\in D_2$ X_{d_2} is a subset of D_1. Then there exist $A_1{}^f\in\mathfrak{L}$ and $B_1{}^g\in\mathcal{R}$ where

$$A_1\in\mathfrak{L}_1,\quad f\colon A_1\to\mathfrak{L}_2$$

$$B_1\in\mathcal{R}_1\quad\text{and}\quad g\colon B_1\to\mathcal{R}_2\,.$$

s.t.

either $$\forall d_1\in A_1\quad \forall d_2\in f(d_1)\,(d_1\in X_{d_2})$$

or $$\forall d_1\in B_1\quad \forall d_2\in g(d_1)\,(d_1\in D_1-X_{d_2})$$

Proof. Define $\tilde{X}=\{\langle d_1, d_2\rangle|d_2\in D_2\wedge d_1\in X_{d_2}\}$. Then there exist $A\in\mathfrak{L}$ and $B\in\mathcal{R}$ s.t. either $A\subseteq\tilde{X}$ or $B\subset D-\tilde{X}$. Let $A=A_1{}^f$ and $B=B_1{}^g$.

$$A\subset\tilde{X}\text{ iff }\forall d_1\in D_1\quad\forall d_2\in D_2\,(\langle d_1, d_2\rangle\in A\supset\langle d_1, d_2\rangle\in\tilde{X})$$

$$\text{iff }\forall d_1\in A_1\quad\forall d_2\in f(d_1)\,(d_1\in X_{d_2})\,.$$

Similarly,

$$B\subseteq D-\tilde{X}\text{ iff }\forall d_1\in D_1\quad\forall d_2\in D_2\,(\langle d_1, d_2\rangle\in B\supset\langle d_1, d_2\rangle\notin\tilde{X})$$

$$\text{iff }\forall d_1\in B_1\quad\forall d_2\in g(d_1)\,(d_1\notin X_{d_2})$$

Remark. The conclusion of this proposition is a generalization of being normal of $(\mathfrak{L}_1, \mathcal{R}_1)$, i.e. being normal is the special case of this proposition that D_2 consists of a single point and $\mathfrak{L}_2=\mathcal{R}_2=\{D_2\}$.

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