solutions manual

for

Real and Complex Analysis, by C. Apelian and S. Surace

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This document contains brief hints and solutions to selected exercises in *Real and Complex Analysis*, by C. Apelian and S. Surace. Solutions are enclosed in the symbols •─○ and ○─•, and a copy of the corresponding exercise is provided for the convenience of the reader. Comments are welcome; please email to capelian@drew.edu.

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chapter 1

▶ [1.1] : Prove properties a) through h) using order properties 1 through 4 and the field properties.

●–○ First of all, the properties to prove are (cf. Ex. 1.1 in the book: they are listed immediately before the problem):

a) $x < y$ if and only if $0 < y - x$.
b) If $x < y$ and $w < z$, then $x + w < y + z$.
c) If $0 < x$ and $0 < y$, then $0 < x + y$.
d) If $0 < z$ and $x < y$, then $xz < yz$.
e) If $z < 0$ and $x < y$, then $yz < xz$.
f) If $0 < x$ then $0 < x^{-1}$.
g) If $x \neq 0$ then $0 < x^2$.
h) If $x < y$, then $x < \frac{(x+y)}{2} < y$.

Here are the proofs:

a) If $x < y$, then add $-x$ to both sides to get $0 < y - x$. If $0 < y - x$, add x to each side to get $x < y$.
b) $x + w < y + w < y + z$ and use transitivity.
c) Apply b) with $x, w = 0$, y and z being replaced by respectively, x, y.
d) By a), $y - x > 0$, which means $z(y - x) > 0$ by Axiom 4. Therefore $zy - zx > 0$, or (using a) again) $yz > xz$.
e) $xz - yz = (x - y)(z) = (y - x)(-z)$. Now, $y - x > 0$ by a), and $-z > 0$, applying a) ($-z < 0$ if and only if $0 < 0 - (-z) = z$). Therefore $(y - x)(-z) > 0$ by Axiom 4. This implies $xz - yz > 0$, which means $xz > yz$ by a) again.
f) Clearly $x^{-1} \neq 0$ (since $xx^{-1} = 1 \neq 0$). If $x^{-1} < 0$, then $1 = xx^{-1} < 0$ (using d)). But if $1 < 0$, then $-1 > 0$ and Axiom 4 yields $1 = (-1)(-1) > 0$. This is a contradiction.
g) If $x > 0$ then Axiom 4 shows that $x^2 > 0$. If $x < 0$ then $-x > 0$ (by (i)) and $x^2 = (-x)^2 > 0$ by Axiom 4.
h) Clearly $2x = x + x < x + y < y + y = 2y$. Multiply each side by $\frac{1}{2}$. Note that $\frac{1}{2} = 2^{-1} = (1 + 1)^{-1} > 0$ since $1 > 0$ as was mentioned in the proof of f). The property drops out.

○—●

▶ $[1.2]$: A *maximal element* or *maximum* of a set $A \subset \mathbb{R}$ is an element $x \in A$ such that $a \leq x$ for all $a \in A$. Likewise, a *minimal element* or *minimum* of a set A is an element $y \in A$ such that $y \leq a$ for all $a \in A$. When they exist, we will denote a maximal element of a set A by $max\, A$, and a minimal element of a set A by $min\, A$. Can you see why a set $A \subset \mathbb{R}$ need not have a maximal or minimal element? If either a maximal or a minimal element exists for $A \subset \mathbb{R}$, is it unique?

●—○ $\mathbb{R}$ has neither a minimal element nor a maximal element. Uniqueness follows since if $M_1 \in S$ and $M_2 \in S$ are maximal elements of some set S, then $M_1 \geq M_2$ (since $M_2 \in S$ and M_1 is maximal) and $M_2 \geq M_1$ (since $M_1 \in S$ and M_2 is maximal). Thus $M_1 = M_2$. The case of a minimal element is analogous and is left to the reader. ○—●

▶ $[1.3]$: A **partition** of a set S is a collection of nonempty subsets $\{A_\alpha\}$ of S satisfying $\bigcup A_\alpha = S$ and $A_\alpha \cap A_\beta = \varnothing$ for $\alpha \neq \beta$. Prove that any two sets $A, B \subset \mathbb{R}$ satisfying the Dedekind completeness property also form a partition of $\mathbb{R}$.

●—○ We need to check that $A \cup B = \mathbb{R}$ and $A \cap B = \varnothing$. The first is property (i) of the Dedekind completeness property in RC. If $x \in A \cap B$, then $x \in A$ and $x \in B$, so $x < x$ by property (ii) of the Dedekind completeness property, contradiction. ○—●

▶ $[1.4]$: In part b) of the above example, show that the set A does not have a maximal element.

●—○ If $x \in A$, then

$$\xi \equiv \frac{x + \frac{1}{2}}{2}$$

belongs to A but (prove!) $\xi > x$, cf. the final part of Ex. 1.1.

Thus x is not a maximal element of A. In fact, the number ξ constructed above is just the mean of x and $\frac{1}{2}$. ○—●

▶ $[1.5]$: Establish the following:

a) Show that, for any positive real number x, there exists a natural number n such that $0 < \frac{1}{n} < x$. This shows that there are rational numbers arbitrarily close to zero on the real line. It also shows that there are real numbers between 0 and x.
b) Show that, for any real numbers x, y with x positive, there exists a natural number n such that $y < nx$.

●—○ There exists $n \in \mathbb{N}$ with $n > \frac{1}{x} > 0$, and invert. (Here we need $x > 0$.) This proves the first part. For the second, apply the Archimedean Property to $\frac{y}{x}$. We get a number $n' \in \mathbb{N}$ with $\frac{y}{x} < n'$, so $y < n'x$. ○—●

▶ $[1.6]$: Use the Archimedean property and the well-ordered property to show that, for any real number x, there exists an integer n such that $n - 1 \leq x < n$. To do this, consider the following cases in order: $x \in \mathbb{Z}$, $\{x \in \mathbb{R} : x > 1\}$, $\{x \in \mathbb{R} : x \geq 0\}$, $\{x \in \mathbb{R} : x < 0\}$.

●—○ The case of $x \in \mathbb{Z}$ is trivial (take $n = x + 1$). If $x > 1$, there exists a positive integer n' with $x < n'$. Let $S = \{m \in \mathbb{N} : m > x\}$. Since $n' \in S$,

$S \neq \varnothing$. By the Well-Ordered Property, one can choose $n \in S$ so that it is the *least* element of S.

I claim that $n - 1 \leq x < n$. The right inequality is the definition of S. Suppose the left inequality was false; then $n - 1 > x$ s o $n - 1 \in S$. But then n would not be the smallest element of S, contradiction. The claim is proved.

The case $x \geq 0$ now follows finding a natural number n as above such that $n - 1 \leq 2 + x < n$, then subtracting two from both sides. The case $x < 0$ can be handled similarly. Choose $M \in \mathbb{N}$ with $-M < x$ (justify this!) and apply the above argument to $x + M$, then subtract M.

⧟

▶ $[\mathbf{1.7}]$: Show that if ξ is irrational and $q \neq 0$ is rational, then $q\xi$ is irrational.

⧟ If $q\xi \in \mathbb{Q}$, then $\xi = q^{-1}(q\xi) \in \mathbb{Q}$, contradiction. ⧟

▶ $[\mathbf{1.8}]$: In this exercise, we will show that for any two real numbers x and y satisfying $x < y$, there exists an irrational number ξ satisfying $x < \xi < y$. The result of this exercise implies that there are infinitely many *irrational* numbers between any two real numbers, and that there are *irrational* numbers arbitrarily close to any real number. To begin, consider the case $0 < x < y$, and make use of the previous exercise.

⧟ Suppose first x and y are *rational*. Pick an irrational number, say $\sqrt{2}$. There exists $n \in \mathbb{N}$ so that $\frac{1}{n}\sqrt{2} < y - x$ by an extension of the Archimedean Property (see Exercise 1.5). Then $x + \frac{1}{n}\sqrt{2}$ is the required ξ. If x and y are not both rational, choose rational numbers r_1, r_2 with $x < r_1 < r_2 < y$ (by the density of the rationals, see RC above this exercise) and apply this construction for r_1 and r_2. ⧟

▶ $[\mathbf{1.10}]$: Prove the above corollary. Show also that the conclusion still holds when the condition $|x| < \epsilon$ is replaced by $|x| \leq \epsilon$.

⧟ If $x \neq 0$, then $|x| > 0$. Set $\epsilon = |x|$ to get $|x| < |x|$, a contradiction. For the second statement, do the same but with $\epsilon = \frac{|x|}{2}$. ⧟

▶ $[\mathbf{1.12}]$: Finish the proof of the triangle inequality. That is, show that

$$||x| - |y|| \leq |x - y| \leq |x| + |y| \text{ for all } x, y \in \mathbb{R}.$$

⧟ First, we prove the rest of part a). Just switch y with $-y$. In detail, we have already shown that

$$|x + y| \leq |x| + |y|, \tag{1}$$

but we must show

$$|x - y| \leq |x| + |y|. \tag{2}$$

Now, in (1), simply replace y by $-y$. Since $|-y| = |y|$, we find (2) follows directly. The rest of part b) may be proved the same way (replacing y with $-y$). ⧟

▶ $[\mathbf{1.13}]$: Show that $|x - z| \leq |x - y| + |y - z|$ for all x, y, and $z \in \mathbb{R}$.

•–∘ This form of the triangle inequality is frequently used. To prove it, note that

$$|x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z|,$$

where we have used the usual triangle inequality with $x - y$ for x and $y - z$ for y. ∘–•

▶ $[\mathbf{1.14}]$: Prove the above corollary.

•–∘ We need to prove

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|. \tag{3}$$

Induction on n. It is true trivially for $n = 1$. Assume it for $n - 1$. Then by the Triangle Inequality

$$|x_1 + \cdots + x_n| \leq |x_1 + \cdots + x_{n-1}| + |x_n|,$$

and applying the case $n - 1$ of (3) concludes the proof. Incidentally, the reader should already be familiar with such inductive arguments. ∘–•

▶ $[\mathbf{1.15}]$: Prove the above properties.

•–∘ All these follow immediately from the corresponding properties in $\mathbb{R}$. We only do a sampling.

(1)

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1, \ldots, x_k) + (y_1, \ldots, y_k) \\
&= (x_1 + y_1, \ldots, x_k + y_k) \\
&= (y_1 + x_1, \ldots, y_k + x_k) \\
&= \mathbf{y} + \mathbf{x}.
\end{aligned}$$

(2)

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= ((x_1, \ldots, x_k) + (y_1, \ldots, y_k)) + (z_1, \ldots, z_k) \\
&= (x_1 + y_1, \ldots, x_k + y_k) + (z_1, \ldots, z_k) \\
&= (x_1 + y_1 + z_1, \ldots, x_k + y_k + z_k) \\
&= (x_1, \ldots, x_k) + (y_1 + z_1, \ldots, y_k + z_k) \\
&= (x_1, \ldots, x_k) + ((y_1, \ldots, y_k) + (z_1, \ldots, z_k)) \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z}).
\end{aligned}$$

9. We only show the first equality here.

$$\begin{aligned}
c(\mathbf{x} + \mathbf{y}) &= c\left((x_1, \ldots, x_k) + (y_1, \ldots, y_k)\right) \qquad = c(x_1 + y_1, \ldots, x_k + y_k) \\
&= (c(x_1 + y_1), \ldots, c(x_k + y_k)) \\
&= (cx_1 + cy_1, \ldots, cx_k + cy_k) \\
&= (cx_1, \ldots, cx_k) + (cy_1, \ldots, cy_k) \\
&= c\mathbf{x} + c\mathbf{y}.
\end{aligned}$$

∘–•

▶ $[1.17]$: Suppose $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{R}^k$. Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary elements of $\mathbb{R}^k$, and c any real number. Establish the following.

a) $c\langle\mathbf{x},\mathbf{y}\rangle = \langle c\mathbf{x},\mathbf{y}\rangle$ b) $\langle\mathbf{x}+\mathbf{y},\mathbf{z}\rangle = \langle\mathbf{x},\mathbf{z}\rangle + \langle\mathbf{y},\mathbf{z}\rangle$

•–○ For a), $c\langle\mathbf{x},\mathbf{y}\rangle = c\langle\mathbf{y},\mathbf{x}\rangle = \langle\mathbf{y},\mathbf{cx}\rangle = \langle\mathbf{cx},\mathbf{y}\rangle$ where Condition 3 of Definition 2.1 was used twice. The proof of b) is similar: $\langle\mathbf{x}+\mathbf{y},\mathbf{z}\rangle = \langle\mathbf{z},\mathbf{x}+\mathbf{y}\rangle = \langle\mathbf{z},\mathbf{x}\rangle + \langle\mathbf{z},\mathbf{y}\rangle = \langle\mathbf{x},\mathbf{z}\rangle + \langle\mathbf{x},\mathbf{y}\rangle$. ○–•

▶ $[1.19]$: Verify the other properties of Definition 2.1 for the dot product.

•–○ $\mathbf{x}\cdot\mathbf{x} = \sum_{j=1}^k x_j^2 \geq 0$ and that sum is clearly positive unless all $x_j = 0$, i.e., $\mathbf{x} = 0$. This proves 1. Property 2 is clear:

$$\mathbf{x}\cdot\mathbf{y} = \sum_{j=1}^k x_j y_j = \sum_{j=1}^k y_j x_j = \mathbf{y}\cdot\mathbf{x}.$$

For Property 3,

$$\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = \sum_{j=1}^k x_j(y_j+z_j) = \sum x_j y_j + \sum x_j z_j = \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{z}.$$

We leave Property 4 to the reader. ○–•

▶ $[1.20]$: For a given norm $|\cdot|$ on $\mathbb{R}^k$ and any two elements $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$, establish

$$|\mathbf{x}-\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|,$$

and

$$\Big|\,|\mathbf{x}| - |\mathbf{y}|\Big| \leq |\mathbf{x}\pm\mathbf{y}|.$$

These results together with part 3 of Definition 2.4 are known as *the triangle inequality* and *the reverse triangle inequality,* as in the case for $\mathbb{R}$ given by Theorem 1.5 on page 9.

•–○ $|\mathbf{x}-\mathbf{y}| = |\mathbf{x}+-\mathbf{y}| \leq |\mathbf{x}|+|-\mathbf{y}| = |\mathbf{x}|+|-1||\mathbf{y}| = |\mathbf{x}+\mathbf{y}|$. For the Reverse Triangle Inequality, repeat the proof given in RC for the regular absolute value but now putting in norms. We shall not rewrite it here, since no new techniques whatsoever are used. ○–•

▶ $[1.21]$: Suppose $\mathbf{x}\in\mathbb{R}^k$ satisfies $|\mathbf{x}| < \epsilon$ for all $\epsilon > 0$. Show that $\mathbf{x} = \mathbf{0}$.

•–○ If $\mathbf{x} \neq 0$, then $|\mathbf{x}| > 0$. Taking $\epsilon = |\mathbf{x}|$ and using the hypothesis of the theorem gives

$$|\mathbf{x}| < |\mathbf{x}|\,,$$

contradiction. ○–•

▶ $[1.22]$: Verify the other norm properties for the induced norm.

•–○ In RC, the Triangle Inequality (i.e. the third property for norms) was proved. For Property 1, note that $|\mathbf{x}| = \sqrt{\langle\mathbf{x},\mathbf{x}\rangle} \geq 0$, with equality if and only if $\langle\mathbf{x},\mathbf{x}\rangle = 0$, i.e. if $\mathbf{x} = 0$ by inner product properties. For Property 2,

$$|c\mathbf{x}| = \sqrt{\langle c\mathbf{x},c\mathbf{x}\rangle} = \sqrt{c^2\,\langle\mathbf{x},\mathbf{x}\rangle} = \sqrt{c^2}\sqrt{\langle\mathbf{x},\mathbf{x}\rangle} = |c|\,|\mathbf{x}|\,.$$

○–•

▶ **[1.24]** : Establish the above properties by using the definitions.

●–○ 1, 2, 3, and 4 are trivial—see the definition of addition and use the corresponding properties of real addition. As an example, we prove Property 1:

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \\ &= (x_2 + x_1) + i(y_2 + y_1) \\ &= (x_2 + iy_2) + (x_1 + iy_1) \\ &= z_2 + z_1. \end{aligned}$$

For 5, note that

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2) = (x_2 + iy_2)(x_1 + iy_1),$$

as the definition is symmetric in both factors. 6 is a messy but straightforward computation, which we shall not write out. For Property 7, note that

$$1z = (1 + i0)(x + iy) = x - 0y + i(0x + y) = x + iy = z.$$

8 is discussed in Exercise 1.25 in more detail. 9 is left to the reader—it is simply a computation. The reader familiar with abstract algebra can define $\mathbb{C} \equiv \mathbb{R}[X]/(X^2 + 1)$ and will get the same result with all these properties becoming obvious consequences of those of $\mathbb{R}$. ○–●

▶ **[1.25]** : Verify the above claim, and that $zz^{-1} = 1$.

●–○ Let $z = a + ib$.

$$zz^{-1} = (a+ib)\left(\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}\right) = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} - i\frac{ab}{a^2+b^2} + i\frac{ba}{a^2+b^2} = 1$$

○–●

▶ **[1.26]** : Suppose z and w are elements of $\mathbb{C}$. For $n \in \mathbb{Z}$, establish the following:

a) $(zw)^n = z^n w^n$ b) $(z/w)^n = z^n/w^n$ for $w \neq 0$

●–○ We only give the proof of a). Induction on n. Assume a) true for $n - 1$. (It is trivial for $n = 1$.) Then

$$(zw)^n = (zw)^{n-1}(zw) = z^{n-1}w^{n-1}zw = z^{n-1}zw^{n-1}w = z^n w^n.$$

We have liberally used commutativity and associativity. This establishes the inductive step: if a) is true for $n - 1$, it is true for n. The proof is complete. ○–●

▶ **[1.27]** : Show that any ordering on $\mathbb{C}$ which satisfies properties (2) on page 5 cannot satisfy properties (1.1) on page 5. (Hint: Exploit the fact that i^2 = -1.)

●–○ Suppose $i > 0$. Then $-1 = i \times i > 0$, or $1 < 0$. But then, multiplying $-1 > 0$ by 1 (and reversing the direction, since $1 < 0$) yields $-1 < 0$, a contradiction.

Thus $i < 0$, or $-i > 0$. Then $-1 = (-i)(-i) > 0$ which yields a contradiction again. ⧟

▶ $[\mathbf{1.29}]$: Prove each of the above properties.

⧟ Properties a)-d) are left to the reader. For e),

$$\overline{(x_1 + iy_1)(x_2 + iy_2)} = x_1 x_2 - y_1 y_2 - i(x_1 y_2 + y_1 x_2),$$

which is the same as

$$\overline{(x_1 + iy_1)} \times \overline{(x_2 + iy_2)} = (x_1 - iy_1)(x_2 - iy_2) = x_1 x_2 - y_1 y_2 - i(x_1 y_2 + y_1 x_2).$$

For f), it is sufficient to treat the case $z_1 = 1$, in view of e). (Why?) Note that

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2},$$

and inspection shows that f holds. These properties state that conjugation is an *automorphism* of $\mathbb{C}$.[1] ⧟

▶ $[\mathbf{1.30}]$: Verify the equivalence of (1.6) with (1.5).

⧟ If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2} = \sqrt{(x + iy)(x - iy)} = \sqrt{z\overline{z}}$. ⧟

▶ $[\mathbf{1.31}]$: Prove the properties in the above theorem.

⧟ We write $z = x + iy$. The expression $|z| = \sqrt{x^2 + y^2}$ makes Properties a)-c) immediate. For property d), note that

$$|z_1 z_2| = \sqrt{z_1 z_2 \overline{z_1}\,\overline{z_2}} = \sqrt{z_1 \overline{z_1} z_2 \overline{z_2}} = \sqrt{z_1 \overline{z_1}}\sqrt{z_2 \overline{z_2}} = |z_1|\,|z_2|\,.$$

Property e) follows from Property d) together with the identity $\left|\frac{1}{z}\right| = \frac{1}{|z|}$, which in turn follows by multiplying both sides by $|z|$ and using d). f) is clear from $|z| = \sqrt{x^2 + y^2}$. g) is clear from the definition. h) can be verified by direct multiplication:

$$z \times \frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1,$$

so $\frac{\overline{z}}{|z|^2} = \frac{1}{z}$. i) is straightforward; just check the definitions $\overline{z} = x - iy$, so $z + \overline{z} = x + iy + x - iy = 2x = 2Re(z)$ and similarly for the imaginary part. The first part of j) follows from $|Re(z)| = |x| = \sqrt{x^2} \le \sqrt{x^2 + y^2} = |z|$, and the second part is similar. ⧟

▶ $[\mathbf{1.34}]$: Show that if θ and ϕ are both elements of $\arg z$ for some nonzero $z \in \mathbb{C}$, then $\phi = \theta + 2\pi k$ for some $k \in \mathbb{Z}$.

⧟ By assumption, we have $re^{i\theta} = re^{i\phi}$, where $r = |z|$. Dividng out, we see that this means $\cos\theta = \cos\phi$ and $\sin\theta = \sin\phi$. These two equalities, together with the basic properties of the trigonometric functions, show that θ and ϕ differ by a multiple of 2π. In fact, we may without loss of generality assume (by subtracting multiples of 2π) that both $\theta, \phi \in [0, 2\pi)$; then we must show

[1] Technically, one must also show it is also one-to-one and onto, but that is straightforward.

$\theta = \phi$. This follows from the basic properties of the trigonometric functions. If $\theta \neq \phi$, then θ and ϕ dtermine different angles from the positive x-axis since $\theta, \phi \in [0, 2\pi)$. So mark the points P_1, P_2 on the unit circle with angles θ and ϕ from the positive x-axis. These points P_1, P_2 must differ in one of their coordinates, which means that either $\sin\theta \neq \sin\phi$ or $\cos\theta \neq \cos\phi$. ○—●

▶ $[\mathbf{1.35}]$: Prove the remaining properties in the above result.

●—○ e) follows from the previous exercise's argument. f) follows from $\sin^2\theta + \cos^2\theta = 1$, since

$$\left|e^{i\theta}\right| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

g) follows from the evenness of $\cos\theta$ and the oddness of $\sin\theta$. Precisely, we have

$$\begin{aligned}\overline{e^{i\theta}} &= \overline{\cos\theta + i\sin\theta}\\ &= \cos\theta - i\sin\theta\\ &= \cos(-\theta) + i\sin(-\theta)\\ &= e^{-i\theta}.\end{aligned}$$

f) is a consequence of g) and the properties of the conjugate:

$$e^{i\theta} + e^{-i\theta} = e^{i\theta} + \overline{e^{i\theta}} = 2Re(e^{i\theta}) = 2\cos\theta.$$

○—●

▶ $[\mathbf{1.39}]$: Show that every quadratic equation of the form $a\,z^2 + b\,z + c = 0$ where $a, b, c \in \mathbb{C}$ has at least one solution in $\mathbb{C}$. Also show that if $b^2 - 4\,a\,c \neq 0$, then the equation has two solutions in $\mathbb{C}$.

●—○ Substitute in the elementary quadratic formula, using the fact that $b^2 - 4ac$ has two square roots in $\mathbb{C}$ if it is not zero. If $b^2 - 4ac = 0$, there will be only one solution. It is left to the reader to see that the elementary quadratic formula is still valid over $\mathbb{C}$.[2] ○—●

▶ $[\mathbf{1.40}]$: Show that for $z_0 = x_0 + iy_0 \in \mathbb{C}$, the product $z_0 z$ can be computed via the matrix product $\begin{bmatrix} x_0 & -y_0 \\ y_0 & x_0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ for any $z = x + iy \in \mathbb{C}$.

●—○ Computation.

$$\begin{bmatrix} x_0 & -y_0 \\ y_0 & x_0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0x - y_0y \\ y_0x + x_0y \end{bmatrix}.$$

Also,

$$(x_0 + iy_0)(x + iy) = x_0x - y_0y + i(y_0x + x_0y).$$

○—●

▶ $[\mathbf{1.41}]$: Verify that any matrix from $\mathcal{M}$ represents a scaling and rotation of the vector it multiplies.

[2] Actually, it is valid in any field of characteristic $\neq 2$ (i.e. where $2 \neq 0$).

We can write any matrix in $\mathcal{M}$ in the form

$$\sqrt{x_0^2+y_0^2}\begin{bmatrix} \frac{x_0}{\sqrt{x_0^2+y_0^2}} & -\frac{y_0}{\sqrt{x_0^2+y_0^2}} \\ \frac{y_0}{\sqrt{x_0^2+y_0^2}} & \frac{x_0}{\sqrt{x_0^2+y_0^2}} \end{bmatrix}.$$

For any elements $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$, we can find an angle θ with $a = \cos\theta, b = \sin\theta$.[3] We apply that to $\frac{x_0}{\sqrt{x_0^2+y_0^2}}$ and $\frac{y_0}{\sqrt{x_0^2+y_0^2}}$ to write the above matrix in the form with $r > 0$

$$r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The expression there is a scaling (by r) times a rotation (by θ).

[3] Proof: Write $\theta = \cos^{-1}(a)$; then $\sin\theta = \pm b$ and if $\sin\theta = -b$ replace θ by $2\pi - \theta$.

chapter 2

▶ **[2.1]** : Determine whether the given set is bounded or unbounded.

a) $S = \left\{ z \in \mathbb{C} : |z| + |z-1| < |z-2| \right\}$

b) $S = \left\{ z \in \mathbb{C} : |z| + |z-1| > |z-2| \right\}$

c) $S = \left\{ \mathbf{x}_n \in \mathbb{R}^3 : \mathbf{x}_n = \left(n^2 2^{-n}, \frac{n^2+1}{2n^3+1}, \frac{\cos n}{\sqrt{n}} \right) \right\}$

a) Bounded. $z \in S$ implies

$$2|z| - 1 \leq |z| + |z-1| < |z-2| < |z| + 2$$

by, in order, the reverse triangle inequality, the definition of S, and the triangle inequality. Thus for $z \in S$:

$$|z| < 3.$$

Hence S is bounded.

b) Unbounded: all sufficiently large real numbers belong to S. In fact, if $x \geq 2$, the condition becomes

$$x + x - 1 > x - 2,$$

which is clearly true for such x. So all $x \geq 2$ are in S, which cannot therefore be bounded.

c) Bounded: each component is bounded. First, $n^2 2^{-n} \leq 2$ for all n. To see this, use induction: it is true for $n = 1, 2, 3$ obviously. If it is true for $n-1$, then $(n-1)^2 \leq 2 \times 2^{n-1}$. But since $n^2 \leq 2(n-1)^2$ if $n > 3$ (Why?) which means that $n^2 \leq 2 \times 2 \times 2^{n-1} = 2 \times 2^n$. We have thus proved that $n^2 2^{-n} \leq 2$ for any n.

Now we look at the other components. The second components are bounded by 1, and so are the third components. This is easy to see as $n^2 + 1 \leq 2n^3 + 1$ and $|\cos n| \leq 1$.

Now for any vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, we have by the Triangle Inequality

$$|\mathbf{x}| = |x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)| \leq |x_1| + |x_2| + |x_3|.$$

So if $x \in S$, then

$|x| \leq 2+1+1 = 4$ by the bounds above on each coordinate.

Hence S is bounded.

▶ $[\mathbf{2.2}]$: Consider a set $S \subset \mathbb{X}$.

a) If S is bounded, is S^C bounded or unbounded?

b) If S is unbounded, is S^C bounded or unbounded?

a) Unbounded. If $x \in S$ implies $|x| \leq M$ (where M is a fixed large real number), then all x with $|x| > M$ belong to S^C. So all vectors whose norm is sufficiently *large* belong to S^C. It is now easy to see that S^C cannot be bounded.

b) This can go either way. If $S = \mathbb{X}$ then $S^C = \varnothing$ is clearly bounded; if $S \subset \mathbb{R}$ is taken as $S = \{x : x \leq 0\}$, then $S^C = \{x : x > 0\}$ is unbounded.

▶ $[\mathbf{2.3}]$: Suppose $\{S_\alpha\} \subset \mathbb{X}$ is a collection of bounded sets.

a) Is $\bigcup S_\alpha$ bounded? b) Is $\bigcap S_\alpha$ bounded?

a) In general, no. Let $S_n = \{n\}$ for $n \in \mathbb{N}$. Then each S_n is bounded (as a one-point set) but $\bigcup_n S_n = \mathbb{N}$ is unbounded by the Archimedean property. If the collection is *finite*, the answer becomes always yes. We leave that to the reader.

b) Yes. Pick any set S_β. Then $S_\beta \subset \{x : |x| \leq M\}$ for some large M, by definition of boundedness. Then

$$\bigcap_\alpha S_\alpha \subset S_\beta \subset \{x : |x| \leq M\}.$$

In particular, $\bigcap_\alpha S_\alpha$ is bounded with the same bound M.

▶ $[\mathbf{2.4}]$: Prove that a set of real numbers S is both bounded above and bounded below if and only if it is bounded.

If S is bounded, say $x \in S$ implies $|x| \leq M$, then $x \in S$ implies $-M \leq x \leq M$. So $-M$ is a lower bound and M an upper bound. Conversely, if we are given a lower bound L and an upper bound U to S, then any $x \in S$ by assumption satisfies $L \leq x \leq U$. If we take $M \equiv \max(|L|, |U|)$, then $x \in S$ implies

$$|x| \leq M,$$

so S is bounded.

▶ $[\mathbf{2.6}]$: Complete the proof of the above theorem. That is, suppose S is a nonempty set of real numbers which is bounded below, and show from this that $\inf S$ exists.

One can repeat the proof of the theorem with a slight modification, but there is a quicker route. If S is bounded below, then $-S \equiv \{x : -x \in S\}$

is bounded *above*, as is easily seen.[1] $-S$ therefore has a least upper bound M. Set $m = -M$; then if $x \in S$, $-x \in -S$. Thus $-x \leq M$ which means $x \geq -M = m$. This shows that m is a lower bound for S. If $\epsilon > 0$, then $M - \epsilon$ is not an upper bound for $-S$; equivalently, $m + \epsilon$ is not a lower bound for S. This shows that m is the *greatest lower bound*. (Why?)

For convenience, we present a modified version of the proof in the text. Let A be the set of all lower bounds on S, and let $B = A^C$. Then $\mathbb{R} = A \cup B$, and $a < b$ for all $a \in A$ and for all $b \in B$. (Why?) Since A and B satisfy our Dedekind completeness property criteria, either A has a maximal element or B has a minimal element. Suppose B has a minimal element, β. Then $\beta \in B$, so β is not an lower bound for S, i.e., there exists $s \in S$ such that $\beta > s$. Since this is a strict inequality, there must exist a real number between β and s. In fact, $c \equiv (\beta + s)/2$ is such a real number, and so $\beta > c > s$. But $c > s$ implies $c \notin A$, i.e., $c \in B$. Since $\beta > c$, we have a contradiction, since $\beta \in B$ was assumed to be the smallest element of B. Therefore, since our completeness criteria are satisfied, and B does not have a minimal element, it must be true that A has a maximal element, call it α. Clearly α is the greatest bound or $\inf S$. ○—●

▶ **[2.8]** : Recall that a *maximal* element of a set S is an element $x \in S$ such that $s \leq x$ for all $s \in S$. Likewise, a *minimal* element of a set S is an element $y \in S$ such that $y \leq s$ for all $s \in S$. When they exist, we may denote a maximal element of a set S by $\max S$, and a minimal element of a set S by $\min S$. Show that if S has a maximal element, then $\max S = \sup S$. Likewise, show that if S has a minimal element, then $\min S = \inf S$.

●—○ We prove only the assertion for the maximal element. Let x be a maximal element and M any upper bound. Then $x \leq M$ because in fact $x \in S$. In particular, if M is the *least* upper bound, then $x \leq M = \sup S$. But x is an upper bound for S, by definition. Thus $\sup S \leq x$ since $\sup S$ is the *least* upper bound. Together this shows that $x = \sup S$. ○—●

▶ **[2.9]** : Prove the above proposition.

●—○ We prove only part a) of the proposition. The rest is entirely analogous. If $u < M_S$, then u cannot be an upper bound for S. Thus there must exist some $s \in S$ with $s > u$. Also $s \leq M_S$ since M_S is an upper bound. We are done. ○—●

▶ **[2.10]** : Use Proposition 1.14 to show that if $\sup S$ exists for $S \subset \mathbb{R}$, then $\sup S$ is unique. Do likewise for $\inf S$.

●—○ We prove that the sup is unique. Suppose both A and B are least upper bounds and, say, $A < B$. There exists $s \in S$ so that $A < s \leq B$ by that proposition, so that A is not an upper bound, a contradiction. The reader can modify the above argument for the inf, or alternatively use the trick already referred to of considering the set $-S$. ○—●

[1] If v is a lower bound for S, $-v$ is an upper bound for $-S$.

▶ $[\mathbf{2.11}]$: Recall that any interval of real numbers contains infinitely many rational and infinitely many irrational numbers. Therefore, if x_0 is a real number, then every neighborhood $N_r(x_0)$ centered at x_0 contains infinitely many rational and infinitely many irrational numbers. Let A and B be subsets of $\mathbb{R}^2$ defined as follows:

$$A = \left\{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\right\} \text{ and } B = \left\{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{I}\right\}.$$

That is, A consists of those vectors from $\mathbb{R}^2$ having only rational components, and B consists of those vectors of $\mathbb{R}^2$ having only irrational components. If $\mathbf{x_0}$ is an arbitrary vector in $\mathbb{R}^2$, show that every neighborhood $N_r(\mathbf{x_0})$ centered at $\mathbf{x_0}$ contains infinitely many elements of A and infinitely many elements of B. Investigate the comparable situation in $\mathbb{R}^k$ and in $\mathbb{C}$.

•–◦ Suppose $\mathbf{x}_0 = (x_1, x_2)$. Choose $\xi_1 \in \mathbb{Q}$ with $\xi_1 \in \left(x_1 - \frac{r}{\sqrt{2}}, x_1 + \frac{r}{\sqrt{2}}\right)$, which we know we can do. Choose $\xi_2 \in \mathbb{Q}$ with $\xi_2 \in \left(x_2 - \frac{r}{\sqrt{2}}, x_2 + \frac{r}{\sqrt{2}}\right)$. Clearly $(\xi_1, \xi_2) \in A$, and

$$|(\xi_1, \xi_2) - (x_1, x_2)| < \sqrt{\left(\frac{1}{2} + \frac{1}{2}\right) r^2} = r.$$

The case for B is similar: just choose $\xi_1, \xi_2 \in \mathbb{I}$ instead of $\mathbb{Q}$. Then $\xi_1, \xi_2 \in B \cap N_r(\mathbf{x}_0)$. The case for $\mathbb{R}^k$ is similar too but there are k real numbers ξ_j $(1 \leq j \leq k)$ and $\sqrt{2}$ is replaced by $\sqrt{k}$ in the above proof. The case for $\mathbb{C}$ is no different from that of $\mathbb{R}^2$. ◦–•

▶ $[\mathbf{2.12}]$: Prove the above claim in the following two steps: First, consider an arbitrary point in $\mathbb{X}$ and show that it must satisfy at least one of the above three characterizations. Then, prove that if a point in $\mathbb{X}$ satisfies one of the above three characterizations, it cannot satisfy the other two.

•–◦ We first show that at least one of 1, 2, 3 hold. If 1 and 2 are false, then for *each* neighborhood $N_u(x_0)$,

a) $N_u(x_0)$ is not contained in A, so $N_u(x_0) \cap A^C \neq \varnothing$.
b) $N_u(x_0)$ is not contained in A^C, so $N_u(x_0) \cap A^C \neq \varnothing$.

These are just the negations of 1 and 2, and together they establish 3. So at least one property holds.

We now show they are mutually exclusive.

Suppose 1 holds, i.e. a neighborhood $N_r(x)$ of x is contained in A. Then $x \in A$, so x is not an exterior point of A and 2 cannot hold. Also, x cannot be a boundary point of A because then we would have $N_r(x) \cap A^C \neq \varnothing$, in contradiction to the assumption $N_r(x) \subset A$.

If 2 holds, then there exists $N_s(x) \subset A^C$. Then $x \in A^C$, so no neighborhood of x a fortiori can be contained in A, so x doesn't satisfy 1. Also since $N_s(x) \cap A = \varnothing$, x doesn't satisfy 3 either.

Similarly, if 3 holds, 2 and 1 do not. Let x is a boundary point of A. Suppose x were an interior point of A; we shall establish a contradiction. In this case, there exists a neighborhood $N_t(x) \subset A$, so $N_t(x) \cap A^C = \varnothing$, contradicting the assumption that x is a boundary point. Thus if 3 holds, 1 does not. A similar argument shows that 2 does not. ○—●

▶ $[\mathbf{2.13}]$: Complete part (iii) of the previous example by showing that $N_r(\mathbf{x_3})$ for $r \geq 1$ must contain at least one point in A and at least one point in A^C.

●—○ If $r \geq 1$, then $N_r(\mathbf{x}_3) \supset N_{1/2}(\mathbf{x_3})$. Now $N_{1/2}(\mathbf{x_3})$ contains points from both A and A^C because $\frac{1}{2} < 1$ and by the above Example. A fortiori, the larger set $N_r(\mathbf{x_3})$ must contain points from both A and A^C. ○—●

▶ $[\mathbf{2.14}]$: For $A \subset \mathbb{X}$, show that $\partial A = \partial\left(A^C\right)$.

●—○ $x \in \partial A$ iff[2] every neighborhood of x contains points of A and A^C, i.e. iff each neighborhood contains points of A^C and $(A^C)^C = A$.

Similarly, $x \in \partial A^C$ iff every neighborhood of x contains points of A^C and $\left(A^C\right)^C = A$. Both these conditions are the same. Thus $x \in \partial A$ iff $x \in \partial(A^C)$, i.e. $\partial A = \partial(A^C)$. ○—●

▶ $[\mathbf{2.15}]$: Show that $\partial\mathbb{Q} = \mathbb{R}$. Show the same for $\partial\mathbb{I}$.

●—○ Any neighborhood of any $x \in \mathbb{R}$ contains elements of $\mathbb{Q}$ and $\mathbb{Q}^C = \mathbb{I}$ (an important fact proved in Chapter 1 using the Archimedean property, pages 6–7) so $x \in \partial\mathbb{Q}$. Thus $\partial\mathbb{Q} = \mathbb{R}$. For $\mathbb{I}$, use Exercise 1.8 of Ch. 1 to see that $\partial\mathbb{I} = \partial\mathbb{Q} = \mathbb{R}$. ○—●

▶ $[\mathbf{2.16}]$: Show that, in fact, if x_0 is a limit point of A, then every deleted neighborhood of x_0 contains *infinitely many* points of A.

●—○ If there were a deleted neighborhood N_1' of x_0 containing only finitely many points of A, say $\zeta_1, \ldots, \zeta_k$, then take

$$r \equiv \min_{1 \leq j \leq k} |\zeta_j - x_0| > 0,$$

i.e. so small that $\zeta_j \notin N_r(x_0)$ for all j. Note that $N_r'(x_0) \subset N_1'$ (Why?) so that its only possible intersection with A consists of the points ζ_j, which we know are not in $N_r'(x_0)$. Thus we see that $N_r'(x_0) \cap A = \varnothing$, a contradiction. ○—●

▶ $[\mathbf{2.17}]$: Show that if $x_0 \in \mathbb{X}$ is not a limit point of $A \subset \mathbb{X}$, then x_0 is a limit point of A^C.

●—○ That x_0 is not a limit point of A means that there is a deleted neighborhood N of A containing no points of A, i.e. contained in A^C. So $N \subset A^C$.

Now if N_2 is any deleted neighborhood of x_0, then $N_2 \cap N \neq \varnothing$ (Why?), so $N_2 \cap A^C \neq \varnothing$ as well. Hence, x_0 is a limit point of A^C. ○—●

▶ $[\mathbf{2.18}]$: It is not necessarily true that if x is a boundary point of a subset A of $\mathbb{X}$, then x must be a limit point of A. Give a counterexample.

●—○ Take $A = \{0\} \subset \mathbb{R}$, $x = 0$. x is clearly a boundary point of A but not a limit point because $N_1'(0) \cap A = \varnothing$. ○—●

[2]In these solutions, we abbreviate "if and only if" to "iff," a notation due to Paul Halmos.

▶ $[\mathbf{2.19}]$: Suppose x_0 is an exterior point of $A \subset \mathbb{X}$. Then show that x_0 can never be a limit point of A. But show that x_0 is necessarily a limit point of A^C.

•–◦ By assumption there exists a neighborhood $N_r(x_0) \subset A^C$. A fortiori $N_r'(x_0) \cap A = \varnothing$, so x_0 is not a limit point of A. By exercise 2.17, x_0 is a limit point of A^C. ◦–•

▶ $[\mathbf{2.20}]$: If $A \subset B \subset \mathbb{X}$, show that $A' \subset B'$.

•–◦ If $x \in A'$, then by definition $N_r'(x) \cap A \neq \varnothing$ for all $r > 0$. Since $A \subset B$, we have in fact $N_r'(x) \cap B \neq \varnothing$ for any r, whence $x \in B'$ as well. ◦–•

▶ $[\mathbf{2.22}]$: Prove or disprove: The point $x \in A \subset X$ is not an isolated point of A if and only if x is a limit point of A.

•–◦ x is not an isolated point of A iff there exists no deleted neighborhood $N_r'(x)$ of x such that $A \cap N_r'(x) = \emptyset$. This is simply the negation of the definition.

In other words, x is not an isolated point iff for all deleted neighborhoods $N_r'(x)$ of x,

$$A \cap N_r'(x) \neq \emptyset,$$

i.e. if $x \in A'$. ◦–•

▶ $[\mathbf{2.23}]$: Suppose $S \subset \mathbb{R}$ is bounded and infinite. Let $M_S = \sup S$ and $m_S = \inf S$. Are M_S and m_S always in S? Are M_S and m_S limit points of S? (In both cases, no. Give counterexamples.)

•–◦ If $S = (0, 1)$, neither $M_S = 1$ nor $m_S = 0$ is in A. If $S = \{-1\} \cup (0, 1) \cup \{2\}$, neither $M_S = 2$ nor $m_S = -1$ is a limit point of S (since $N_1'(0) \cap S = \varnothing$ and $N_1'(2) \cap S = \varnothing$). ◦–•

▶ $[\mathbf{2.24}]$: Prove the above proposition.

•–◦ The following statements are equivalent to each other:

(1) G is open
(2) Each $x \in G$ has a neighborhood $N_r(x)$ contained in G (by the definition of openness)
(3) Each $x \in G$ is an interior point of G

That establishes the proposition. ◦–•

▶ $[\mathbf{2.25}]$: For any real $r > 0$ and any $x_0 \in \mathbb{X}$, show that $N_r'(x_0)$ is an open set.

•–◦ If $x \in N_r(x_0)$, $x \neq x_0$, choose ρ as in the last example. Then $N_\rho(x) \subset N_r(x_0)$. We need to shrink ρ so that x_0 does not belong to that neighborhood of x, because then the neighborhood will be contained in $N_r'(x_0)$.

Set $\rho' = \min(\rho, |x - x_0|)$. Then $N_{\rho'}(x) \subset N_r'(x_0)$ because $x_0 \notin N_{\rho'}(x)$. ◦–•

▶ $[\mathbf{2.28}]$: Complete the proof of part (ii) of the above proposition.

•–∘ Choose $r = \min(r_1, r_2)$. Then $N_r(x) = N_{r_1}(x) \cap N_{r_2}(x)$ in fact because that intersection is equal to the set

$$\{x : |x - x_0| \leq r_1, |x - x_0| \leq r_2\} = \{x : |x - x_0| \leq \min(r_1, r_2)\}.$$

∘–•

▶ $[\mathbf{2.29}]$: Generalize part (ii) of the above proposition as follows. Let $G_1, G_2, ..., G_n$ be open sets in $\mathbb{X}$. Prove that $\bigcap_{j=1}^{n} G_j$ is also open in $\mathbb{X}$. The generalization does not hold for arbitrary collections $\{G_\alpha\}$ of open sets. That is, it is not true that $\bigcap_\alpha G_\alpha$ is necessarily open if each G_α is open. Find a counterexample to establish the claim.

•–∘ The assertion about $\bigcap_{j=1}^{n} G_j$ follows by induction: we know it for $n = 1$ (trivially) and if it's true for $n - 1$, $\bigcap_{j=1}^{n-1} G_j$ is open. So by the result just proved

$$\bigcap_{j=1}^{n} G_j = \left(\bigcap_{j=1}^{n-1} G_j \right) \cap G_n \quad \text{is open, QED.}$$

The second assertion is false: take $G_j = \left(-\infty, \frac{1}{j}\right)$. Then $\bigcap_{j=1}^{\infty} G_j = (-\infty, 0]$ which is not open. ∘–•

▶ $[\mathbf{2.30}]$: Show in each of the cases $\mathbb{X} = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^k$ that one can find an open interval I_x such that $I_x \subset N_r(x) \subset G$ as claimed in the above theorem. Also, give the detailed argument justifying the final claim that $G = \bigcup_{x \in G} I_x$.

•–∘ The case of $\mathbb{R}$ is immediate as a neighborhood is an open interval; we take $I_x = N_r(x)$. Fix a neighborhood $N_r(\mathbf{x}) \subset \mathbb{R}^k$ and suppose $\mathbf{x} = (x_1, \ldots, x_k)$. Note that

$$\left(x_1 - \frac{r}{\sqrt{k}}, x_1 + \frac{r}{\sqrt{k}}\right) \times \left(x_2 - \frac{r}{\sqrt{k}}, x_2 + \frac{r}{\sqrt{k}}\right) \times \cdots \times \left(x_k - \frac{r}{\sqrt{k}}, x_k + \frac{r}{\sqrt{k}}\right) \subset N_r(\mathbf{x}),$$

as one sees thus: if $\xi = (\xi_1, \ldots, \xi_k)$ belongs to that product of intervals, then

$$|\xi - \mathbf{x}| = \sqrt{(\xi_1 - x_1)^2 + \cdots + (\xi_k - x_k)^2} < \sqrt{r^2} = r, \quad \text{so } \xi \in N_r(\mathbf{x}).$$

The case of $\mathbb{C}$ is left to the reader.

Finally, to see that $G = \bigcup_{x \in G} I_x$, note that for each $w \in G$, $w \in \bigcup_{x \in G} I_x$ because $w \in I_w$. Thus $G \subset \bigcup_{x \in G} I_x$. Next, since each $I_x \subset G$, $\bigcup_{x \in G} I_x \subset G$. These two assertions prove the claim $G = \bigcup_{x \in G} I_x$. ∘–•

▶ $[\mathbf{2.31}]$: Prove the above proposition.

•–∘ a) follows by inspection, because both $\varnothing = \mathbb{X}^C$ and $\mathbb{X} = \varnothing^C$ are open. For (ii) and (iii) use De Morgan's laws. For instance, let F_1, F_2 be closed. Then

$$(F_1 \cup F_2)^C = F_1^C \cap F_2^C,$$

and since F_1^C, F_2^C are open, so is their intersection. By definition, then $F_1 \cap F_2$ is closed. A similar argument works for c). Let F_α be a collection of closed sets. Then

$$\left(\bigcap_{\alpha} F_\alpha\right)^C = \bigcup_{\alpha} F_\alpha^C,$$

which is an open set because each F_α^C is open. ○—●

▶ $[\mathbf{2.32}]$: Part b) of the above proposition can be generalized in the same way as the corresponding part in Proposition 3.4 as described in a previous exercise. That is, let $F_1, F_2, \ldots, F_n$ be closed sets in $\mathbb{X}$. Prove that $\bigcup_{j=1}^n F_j$ is also open in $\mathbb{X}$. Does the generalization hold for arbitrary collections $\{F_\alpha\}$ of closed sets? That is, is it true that $\bigcup F_\alpha$ is necessarily closed if each F_α is closed? (Answer: See the next exercise.)

●—○ Part b) does work for finite unions by a quick induction using part b), but it does not work for arbitrary collections: take the closed sets $F_n = \left[\frac{1}{n}, 1\right]$ and consider $\bigcup_n F_n = (0, 1]$, which isn't closed. ○—●

▶ $[\mathbf{2.33}]$: Come up with an infinite collection of closed sets $F_1, F_2, F_3, \ldots$ where $\bigcup_{j=1}^\infty F_j$ is not closed.

●—○ See the previous solution for the example $F_n = \left[\frac{1}{n}, 1\right]$. ○—●

▶ $[\mathbf{2.34}]$: Suppose F is a finite set in $\mathbb{X}$. Show that F is closed.

●—○ We need to show that $\mathbb{X} - F$ is open. Now $\mathbb{X} - \{x\}$ is easily seen to be open for any $x \in \mathbb{X}$; it is just the infinite union

$$\bigcup_n N_n'(x).$$

Now $\mathbb{X} - F$ is a finite intersection of sets of the form $\mathbb{X} - \{x\}$, so by an exercise 2.32 $\mathbb{X} - F$ is open, hence F is closed. ○—●

▶ $[\mathbf{2.35}]$: If $F \subset \mathbb{X}$ is closed, and $x_0 \in \mathbb{X}$ is a limit point of F, prove that $x_0 \in F$.

●—○ Let x_0 be a limit point of F. Suppose $x_0 \notin F$. Since F^C is open by definition, there is a neighborhood $N_r(x_0)$ such that $N_r(x_0) \subset F^C$. A fortiori $N_r'(x_0) \subset F^C$. This means that $N_r'(x_0) \cap F = \varnothing$, or x_0 is not a limit point of F, a contradiction. ○—●

▶ $[\mathbf{2.36}]$: Show that for $A \subset \mathbb{X}$, if x is a boundary point of A then $x \in \overline{A}$. Also, if $\sup A$ exists then $\sup A \in \overline{A}$, and if $\inf A$ exists then $\inf A \in \overline{A}$.

●—○ Assume x is a boundary point of A. We must show that $x \in A$ or $x \in A'$. Suppose $x \notin A$. Now, by the definition of boundary points, every neighborhood $N_r(x)$ intersects both A and A^C. All we actually need to use is that every neigborhood $N_r(x)$ intersects A, i.e. $N_r(x) \cap A \neq \varnothing$.

But by assumption $x \notin A$. Therefore we actually have for all $r > 0$

$$N_r'(x) \cap A \neq \varnothing,$$

i.e. x is a limit point of A.

We have thus shown the following: If $x \in \partial A$, either $x \in A$ or $x \in A'$. Thus $\partial A \subset \overline{A} = A \cup A'$.

We now prove that if $A \subset \mathbb{R}$ is bounded above, $\sup A \in \overline{A}$. We leave the proof for the inf to the reader. Suppose $\sup A \notin A$ (because if the sup were in A, our conclusion would follow at once); then for all $r > 0$ there exists $a \in A$ with

$$\sup A - r < a \leq \sup A, \text{ but because } \sup A \notin A, \ \sup A - r < a < \sup A.$$

It follows that $N_r'(\sup A) \cap A \neq \varnothing$ because it contains a. This is true for all r, so $\sup A \in A' \subset \overline{A}$ in fact. ○—●

▶ $[\mathbf{2.37}]$: Show that for any set $A \subset \mathbb{X}$, the point x is in $\overline{A}$ if and only if every neighborhood of x contains a point of A.

●—○ This is an important characterization of the closure. We denote the condition

> (*) Every neighborhood of x intersects A, i.e. if $r > 0$ then $N_r(x) \cap A \neq \varnothing$.

by (*). Suppose $x \in \overline{A} = A \cup A'$. If $x \in A$, then x satisfies (*) because $x \in N_r(x) \cap A$ for all $r > 0$. If $x \in A'$, then for all $r > 0$,

$$N_r(x) \cap A \supset N_r'(x) \cap A \neq \emptyset,$$

by the definition of limit points. In this case too, thus, x satisfies (*). We have thus shown $x \in \overline{A}$ implies x satisfies (*).

Now, let's show the reverse. Suppose x satisfies (*); we show $x \in \overline{A}$. If $x \in A$, the result is immediate, so assume $x \notin A$. Then

$$N_r'(x) \cap A = N_r(x) \cap A \neq \emptyset,$$

by, repsectively, $x \notin A$ and the condition (*). It follows that $x \in A' \subset \overline{A}$. ○—●

▶ $[\mathbf{2.38}]$: If $A \subset B \subset \mathbb{X}$, show that $\overline{A} \subset \overline{B}$.

●—○ By assumption $A \subset B$; by exercise 2.20 $A' \subset B'$. Thus $\overline{A} = A \cup A' \subset B \cup B' = \overline{B}$. ○—●

▶ $[\mathbf{2.40}]$: Complete the proof of part b) of the above proposition.

●—○ We actually sneakily made this entire proposition into an earlier exercise. The solution was given earlier (2.37). ○—●

▶ $[\mathbf{2.41}]$: For a set $S \subset \mathbb{X}$, prove that $\overline{S} = S \cup \partial S$. Is it necessarily true that $S' = \partial S$? (Answer: No.)

●—○ Let $x \in \partial S$; we show $x \in \overline{S}$. By assumption, all neighborhoods of x intersect S (and S^C, but we don't care). Hence $x \in \overline{S}$ by the above proposition. Thus

$$S \cup \partial S \subset \overline{S}.$$

Conversely, let $x \in \overline{S}$. Then all neighborhoods $N_r(x)$ satisfy

$$N_r(x) \cap S \neq \varnothing,$$

so either $x \in S$ (in which case $x \in S \cup \partial S$) or also

$$x \in N_r(x) \cap S^C, \quad \text{all } r.$$

So either $x \in S$ or $x \in \partial S$.

As a counterexample for the last question, let S be a one-point set; then $S' = \varnothing$ but $\partial S = S$. ○—●

▶ $[\mathbf{2.42}]$: For $A, B \subset \mathbb{X}$, prove the following: a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ b) $\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$

●—○ For a) it will be sufficient to show that $(A \cup B)' = A' \cup B'$. The inclusion $\supset$ follows because by 2.20

$$(A \cup B)' \supset A', \quad (A \cup B)' \supset B'.$$

If x is a limit point of $A \cup B$, then we must show conversely that x is a limit point of either A or B. If not there exist deleted neighborhoods $N'_{r_1}(x)$ and $N'_{r_2}(x)$ such that $N'_{r_1}(x) \cap A = \emptyset$ and $N'_{r_2}(x) \cap B = \emptyset$. Set $r = \min(r_1, r_2)$ to get $N'_r(x) \cap (A \cup B) = \emptyset$, a contradiction since $x \in (A \cup B)'$. We have shown that $(A \cup B)' = A' \cup B'$. The first part now follows from $\overline{S} = S \cup S'$.

The same proof will *not* work for arbitrary unions. (Try it!) We do know, however, by Proposition 3.10 that

$$\overline{\bigcup A_\alpha}$$

is the smallest closed set containing $\bigcup A_\alpha$. Similarly, $\overline{A_\alpha}$ is the smallest closed set containing A_α. Then $\overline{\bigcup A_\alpha}$ is closed and contains each A_α (and thus $\overline{A_\alpha}$), hence contains $\bigcup \overline{A_\alpha}$, establishing the inclusion in question. ○—●

▶ $[\mathbf{2.43}]$: For $\{A_\alpha\}$ with each $A_\alpha \subset \mathbb{X}$, find an example where $\bigcup \overline{A_\alpha} \neq \overline{\bigcup A_\alpha}$.

●—○ $A_n = \left(\frac{1}{n}, \infty\right)$ for $n \in \mathbb{N}$. We have

$$\bigcup \overline{A_n} = (0, \infty); \quad \overline{\bigcup A_n} = [0, \infty).$$

○—●

▶ $[\mathbf{2.44}]$: Presuming all sets are from $\mathbb{X}$, how do the following sets compare:

a) $\overline{A \cap B}, \overline{A} \cap \overline{B}$ b) $\bigcap \overline{A_\alpha}, \overline{\bigcap A_\alpha}$

c) $(A \cup B)', A' \cup B'$ d) $(A \cap B)', A' \cap B'$

●—○ In this exercise we shall frequently use the fact that the closure of a set S is the *smallest closed set containing* S, which is Proposition 3.16.

a) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ because the latter is a closed set (Why?) containing $A \cap B$, thus containing $\overline{A \cap B}$. They need not be equal: take $A = \mathbb{Q}$, $B = \mathbb{I}$. Then $\overline{A \cap B} = \varnothing$, but $\overline{A} \cap \overline{B} = \mathbb{R}$.
b) The reasoning is the same: $\bigcap \overline{A_\alpha}$ is a closed set containing $\bigcap A_\alpha$, hence it contains $\overline{\bigcap A_\alpha}$.
c) $(A \cup B)' = A' \cup B'$; see the solution to 2.42.

d) $(A \cap B)' \subset A' \cap B'$ because by 2.20 we have

$$(A \cap B)' \subset A', \quad (A \cap B)' \subset B'.$$

Equality need not hold: take $A = \mathbb{Q}, B = \mathbb{I}$ as above.

▶ $[\mathbf{2.47}]$: Is it true in general that if U is open in $\mathbb{X}$, then for any subset $S \subset \mathbb{X}$ with $U \subset S$ the set U will be open in S? (Yes.)

Yes. If $V = U$, then V is open in $\mathbb{X}$ by assumption and $U = V \cap S$, so Definition 3.17 applies.

▶ $[\mathbf{2.48}]$: Suppose $U \subset S \subset \mathbb{X}$ where S is closed. Is it possible for U to be open in S if U is not open in $\mathbb{X}$? (Yes. Give examples.)

For an example of this, see the preceding example in RC (3.18).

▶ $[\mathbf{2.49}]$: Suppose $U \subset S \subset \mathbb{X}$ where U is open in S, and S is open. Is U necessarily open in $\mathbb{X}$? (Yes.)

By assumption $U = V \cap S$, where V is open in $\mathbb{X}$. But S is also open in $\mathbb{X}$. So $U = V \cap S$ is also open in $\mathbb{X}$ by Proposition 3.4.

▶ $[\mathbf{2.50}]$: Suppose U_1 and U_2 are subsets of $S \subset \mathbb{X}$ and are relatively open in S. Prove the following:

a) $U_1 \cup U_2$ is relatively open in S. b) $U_1 \cap U_2$ is relatively open in S.
c) Generalize a) to infinite unions. d) Show that b) fails for infinite intersections.

a) If U_1, U_2 are relatively open then there exist open sets $V_1, V_2 \subset \mathbb{X}$ so that $U_j = S \cap V_j$ for $j = 1, 2$. Then $U_1 \cup U_2 = S \cap (V_1 \cup V_2)$, and $V_1 \cup V_2$ is open in $\mathbb{X}$. Hence $U_1 \cap U_2$ is relatively open in S.

b) Same proof as a), but now use $U_1 \cap U_2 = (V_1 \cap V_2) \cap S$ and note that $V_1 \cap V_2$ is open in $\mathbb{X}$.

c) Same proof as a) by taking infinite unions of V's.

d) Take, say, $U_n = \left(1, 2 + \frac{1}{n}\right)$ and $S = (0, \infty)$.

▶ $[\mathbf{2.51}]$: Suppose $S \subset \mathbb{X}$. Show that $\varnothing$ and S are both relatively open in S, and that $\varnothing$ and S are both relatively closed in S.

All this follows from the formulas

$$\varnothing = \varnothing \cap S, \quad S = \mathbb{X} \cap S,$$

and using the fact that $\varnothing, \mathbb{X}$ are both open and closed (in $\mathbb{X}$).

▶ $[\mathbf{2.52}]$: Prove the above proposition.

A is dense in B iff $B \subset \overline{A}$ iff $B \subset A \cup \partial A$ iff each $b \in B$ belongs to A or is a boundary point of A. Hence, a) and c) are equivalent. The equivalence of a) and b) is deduced in view of the characterization of the closure in Proposition 3.16.

► $[\mathbf{2.53}]$: Give the detailed answers to the three (Why?) questions in the above proof.

•–○ First question: It is true that if $a_0 \in A$, $b_0 \in B$, then $a_0 \leq b_0$. Take the inf over $b_0 \in B$ to get $a_0 \leq \inf B$. Then take the sup over $a_0 \in A$ to get $\sup A \leq \inf B$. Thus $a \leq b$.

2nd: If $\{\mathcal{I}_n\}$ is a decreasing sequence of closed bounded intervals, then the projections onto each axis form a decreasing sequence of closed bounded intervals too. Those projections are I_{1n} and I_{2n}.

3rd: For any collections of sets $\{A_\alpha\}, \{B_\alpha\}$, it is true that

$$\left(\bigcap_\alpha A_\alpha\right) \times \left(\bigcap_\alpha B_\alpha\right) = \bigcap_\alpha (A_\alpha \times B_\alpha).$$

If (a,b) belongs to the former then $a \in A_\alpha$ for each α, $b \in B_\alpha$ for each α, so $(a,b) \in A_\alpha \times B_\alpha$ for each α. Thus (a,b) belongs to the latter set. Similarly, if (a,b) belongs to the latter set, it belongs to the former. ○–•

► $[\mathbf{2.54}]$: What is the significance of the last example? How about Examples 7.1 and 7.2?

•–○ They show that the hypotheses of closedness and boundedness are necessary. ○–•

► $[\mathbf{2.58}]$: Prove the above corollary.

•–○ Note that $\bigcap \mathcal{F}_n \neq \emptyset$ by the Nested Bounded Closed Sets Theorem. The diameter of this intersection is less than or equal to $r^{n-1} diam(\mathcal{F}_1)$ for all n (Why?) so the diameter is 0, and the intersection can have at most one point. ○–•

► $[\mathbf{2.59}]$: In the above, we argued that for arbitrary $p \in A$, there exists a neighborhood of p such that $N_r(p) \subset \mathcal{O}_{\alpha_p}$, and that there also exists a point $p' \in \mathbb{X}$ having coordinates in $\mathbb{Q}$, that is the center of a neighborhood $N_\beta(p')$ of radius $\beta \in \mathbb{Q}$ such that $p \in N_\beta(p') \subset N_r(p) \subset \mathcal{O}_{\alpha_p}$. Give the detailed argument for the existence of p' and β with the claimed properties.

•–○ We give the proof when $\mathbb{X} = \mathbb{R}$. Fix $p \in \mathbb{R}$ and choose $p' \in \mathbb{Q}$ very close to p, so close that p' and p are both contained in a neighborhood $N_r(p)$ which in turn is so small that $N_{3r}(p)$ is contained in some element $\mathcal{O}_\alpha$ of that covering. (In other words, choose r first, then choose p'.) We can take a rational number β so that $\beta > |p - p'|$, implying $p \in N_\beta(p')$. β is to be taken smaller than r.

The collection of neighborhoods $N_\beta(p')$ is countable since it can be indexed by $\mathbb{Q} \times \mathbb{Q}$ (Why?) and the latter set, as the product of two countable sets, is countable. (To see that, use a diagonal argument.) ○–•

► $[\mathbf{2.60}]$: Is the set given by $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \subset \mathbb{R}$ compact? (Answer: No. Prove it.)

•–○ No. Consider the covering $U_n = (\frac{1}{n} - \frac{1}{100n^2}, \frac{1}{n} + \frac{1}{100n^2})$. There is no finite subcovering since each U_n contains precisely one element of that set. ○–•

▶ $[\mathbf{2.65}]$: Complete the proof of the above proposition for the cases $\mathbb{R}^k$ and $\mathbb{C}$. To get started, consider the case of $\mathbb{R}^2$ as follows. Suppose $E, F \subset \mathbb{R}$ are connected sets. Prove that $E \times F$ is a connected set in $\mathbb{R}^2$. (Hint: Fix a point $(a, b) \in E \times F$, and show that $\{a\} \subset E$ and $\{b\} \subset F$ are connected. Then show that $(\{a\} \times F) \cup (E \times \{b\})$ is connected.) Extend this result to $\mathbb{R}^k$.

•–○ First, we'll show that $\{a\} \times F \cup E \times \{b\}$ is closed. Note that $\{a\} \times F$ and $E \times \{b\}$ are, because otherwise, given a disconnecting partition as in the definition, we could project onto the second or first (respectively) factors to get a partition of F or E, respectively. (A more advanced statement would be that $\{a\} \times F$ is *homeomorphic* to F.) The union is connected by 5.10 since $\{a\} \times F \cap E \times \{b\}$ is nonempty; it contains (a, b).

So, we now fix $a \in E$, and write

$$E \times F = \bigcup_{b \in F} \{a\} \times F \cup E \times \{b\},$$

while

$$\bigcap_{b \in F} \{a\} \times F \cup E \times \{b\} = \{a\} \times F \neq \emptyset.$$

Apply Prop. 5.10 again. ○–•

chapter 3

▶ $[3.1]$: Show that $\lim \left(\frac{1+i}{4}\right)^n = 0$.

The hint is that $\left|\left(\frac{1+i}{4}\right)^n - 0\right| = \left(\frac{\sqrt{2}}{4}\right)^n$. Now use logarithms.

▶ $[3.5]$: Show that $\lim i^n$ does not exist. (Hint: Assume it does.)

Suppose the limit did exist, call it L. Then we can find $N \in \mathbb{N}$ such that for $n > N$, i^n is really close to L, say $|i^n - L| < 1$. By the triangle inequality, it follows that if $n, m > N$, we have $|i^n - i^m| < 2$. Taking $m = n + 2$, we have a contradiction (because, for instance, $|i - (-i)| = 2$).

▶ $[3.6]$: Prove part *2.* of the above proposition.

$\mathbb{C}$ and $\mathbb{R}^2$ are essentially the same if multiplication is ignored, and the idea of taking a limit is the same, so this is essentially included in the theorem.

▶ $[3.8]$: Consider the sequence $\{\mathbf{x}_n\}$ in $\mathbb{R}^k$, where $\mathbf{x}_n = (x_{1n}, x_{2n}, \ldots, x_{kn})$. Show that $\{\mathbf{x}_n\}$ is bounded in $\mathbb{R}^k$ if and only if each sequence $\{x_{jn}\}$ is bounded in $\mathbb{R}$ for $j = 1, 2, \ldots, k$. State and prove the analogous situation for a sequence $\{z_n\} \subset \mathbb{C}$.

This follows from the next fact: If $\mathbf{x}_n = (x_{1n}, \ldots, x_{kn})$, then $|\mathbf{x}_n| \leq |x_{1n}| + \cdots + |x_{kn}|$, so if each of the sequences on the right are bounded by $M_1, \ldots, M_n$, the one on the left is bounded by $M_1 + \cdots + M_n$. Same for $\mathbb{C}$. The converse implication is even easier, because for each k, n, $|x_{kn}| \leq |\mathbf{x}_n|$.

▶ $[3.10]$: Complete the proof of the above proposition.

a) is immediate from b) and the fact that $\lim x_0 = x_0$, which is clear. We now prove c). Suppose $\lim x_n = x$; we show thaat $\lim cx_n = cx$. Fix $\epsilon > 0$. There exists $M \in \mathbb{N}$ such that $n > M$ implies $|x_n - x| < \frac{\epsilon}{|c|+1}$, which implies

$$|cx_n - cx| < \epsilon, \ \forall n > M.$$

This is the definition of $\lim cx_n = cx$. Next, e) follows from the reverse triangle inequality: if $\lim x_n = L$,

$$||x_n| - |L|| \leq |x_n - L|.$$

We can make the right side arbitrarily small for n sufficiently large, and so we can make the left side small for n large enough. ○—●

▶ $[\mathbf{3.11}]$: Show that the converse of property e). is true only when $x = 0$. That is, show the following: If $\lim |x_n| = 0$, then $\lim x_n = 0$. If $\lim |x_n| = x \neq 0$, then it is not necessarily true that $\lim x_n = x$.

●—○ For the first part, fix $\epsilon > 0$. Then we can choose $N \in \mathbb{N}$ large enough that $n > N$ implies

$$|x_n| = ||x_n|| = ||x_n| - 0| < \epsilon,$$

which is precisely the condition for $\lim x_n = 0$.

For the other half, consider the sequence $(-1)^n x$ where $x \neq 0$. ○—●

▶ $[\mathbf{3.12}]$: In the previous proposition, show that property d) is still true when $\{x_n\}$ is a sequence of real numbers while $\{y_n\}$ is a sequence from $\mathbb{R}^k$ or $\mathbb{C}$.

●—○ It is proved precisely the same way as d), except that instead of the Cauchy-Schwarz inequality, one uses the fact that

$$|xy| = |x||y|$$

for $x \in \mathbb{R}$, $y \in \mathbb{R}^k$ or $\mathbb{C}$. ○—●

▶ $[\mathbf{3.13}]$: Complete the proof of the above theorem by handling the case where $z_n \neq 1$ for at least one $n \in \mathbb{N}$.

●—○ If a_n, b_n are two sequences of complex numbers, then

$$\lim a_n b_n = \lim a_n \cdot \lim b_n,$$

by Proposition 2.3d, whenever $\lim a_n, \lim b_n$ exist. So, using the case of Proposition 2.4 already proved, we find:

$$\lim \frac{z_n}{w_n} = \lim z_n \lim \frac{1}{w_n} = z\frac{1}{w} = \frac{z}{w}.$$

○—●

▶ $[\mathbf{3.14}]$: Prove the above corollary.

●—○ Consider $\{y_n - x_n\}$. ○—●

▶ $[\mathbf{3.19}]$: Prove part $b)$ of the above theorem.

●—○ One can modify the proof of part $a)$ appropriately. Or, use the following argument. Replace x_n by $-x_n$, which is a nondecreasing or increasing sequence. It is bounded above because x_n is bounded below. So by part a)

$$\lim -x_n = \sup -x_n = -\inf x_n,$$

which implies

$$\lim x_n = \inf x_n.$$

○—●

▶ $[\mathbf{3.20}]$: Show that if x is a limit point of the sequence $\{x_n\}$ in $\mathbb{X}$, then for any $\epsilon > 0$, the neighborhood $N_\epsilon(x)$ contains infinitely many elements of $\{x_n\}$.

•─○ If only finitely many were contained in the neighborhood $N_\epsilon(x)$, then for n sufficiently large $|x_n - x| \geq \epsilon$, so no subsequence can converge to x. ○─•

► $[\mathbf{3.21}]$: Prove that if a sequence is convergent, the limit of the sequence is the only limit point of that sequence.

•─○ This follows from Prop. 3.3. ○─•

► $[\mathbf{3.23}]$: Consider a set of points $S \subset \mathbb{X}$, and let S' be the set of limit points of S. Show that if $x \in S'$, there exists a sequence $\{x_n\} \subset S$ for $n \geq 1$ such that $\lim x_n = x$.

•─○ Pick $x \in S'$. For each n, choose $x_n \in S$ such that $x_n \in N'_{1/n}(x)$, possible because otherwise x would not be a limit point. Then $|x_n - x| < \frac{1}{n}$. So, clearly $\lim x_n = x$. ○─•

► $[\mathbf{3.24}]$: Suppose $\{x_n\} \subset S \subset \mathbb{X}$ is a sequence convergent to x. Show that $x \in \bar{S}$, but it is not necessarily true that $x \in S'$.

•─○ If a sequence of elements of S converged to x and $x \notin \bar{S}$, there's a neighborhood $N_r(x)$ containing x that doesn't intersect S. (Indeed, we can even arrange $N_r(x) \cap \bar{S} = \emptyset$; this is possible because $\bar{S}$ is closed, so $\bar{S}^C$ is open.) This being so, no sequence in S can converge to x as it would eventually fall in $N_r(x)$. The last part is left to the reader (try, e.g. a one-point set). ○─•

► $[\mathbf{3.25}]$: Suppose $A \subset \mathbb{X}$ is compact and $B \subset \mathbb{X}$ is closed such that $A \cap B = \varnothing$. Then $dist(A, B) \equiv \inf_{\substack{a \in A \\ b \in B}} |a - b| > 0$. To show this, assume $dist(A, B) = 0$. Then there exist sequences $\{a_n\} \subset A$ and $\{b_n\} \subset B$ such that $\lim |a_n - b_n| = 0$. Exploit the compactness of A to derive a contradiction.

•─○ Note that $\{a_n\}$ has a limit point a by the Bolzano-Weierstrass theorem, since A is bounded; this a must lie in A by the previous exercise since A is closed. By extracting a subsequence of $\{a_n\}$ (and extracting the same subsequence from $\{b_n\}$) if necessary, assume $\lim a_n = a \in A$. We will prove that b_n converges to a as well.

Indeed, fix $\epsilon > 0$. We can find $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|a_n - a| < \epsilon/2$. We can find $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|a_n - b_n| < \epsilon/2$. Thus

$$|b_n - a| \leq \epsilon/2 + \epsilon/2 = \epsilon, \quad \forall n > \max(N_1, N_2).$$

This implies that $\lim b_n = a$.

Thus, by the previous exercise again, $a \in B$. Thus $A \cap B \neq \emptyset$, a contradiction. ○─•

► $[\mathbf{3.27}]$: Answer the (Why?) in the above proof, and prove that $\lambda \in \mathcal{L}$.

•─○ For the (Why?) question, choose some limit point $L_2 \in (\Lambda - 1/2, \Lambda + 1/2)$, consider a subsequence converging to L_2, and take a term x_{n_2} in that sequence of very high index (bigger than $n_2 > n_1$) and such that $x_{n_2} \in (\Lambda - 1/2, \Lambda + 1/2)$. For the second part, repeat the above proof while replacing Λ by λ, or replace x_n by $-x_n$ and note that lim sups change into lim infs (proof left to the reader). ○─•

▶ $[\mathbf{3.33}]$: Suppose $\{x_n\}$ is a bounded sequence of real numbers where $x_n \geq 0$ for $n = 1, 2, \ldots$. Suppose also that $\limsup x_n = 0$. Show in this case that $\lim x_n = 0$.

●–○ By Proposition 3.11a, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $n > N$ implies $x_n < \epsilon$. This means that $|x_n| < \epsilon$ since $x_n \geq 0$. Thus $\lim x_n = 0$. ○–●

▶ $[\mathbf{3.35}]$: Suppose $\{x_n\}$ is a sequence in $\mathbb{X}$ such that $|\,x_{n+1} - x_n\,| \leq c\,r^n$ for some positive $c \in \mathbb{R}$ and $0 < r < 1$. Prove that $\{x_n\}$ is a Cauchy sequence, and therefore $\lim x_n = x$ exists.

●–○ Note that for $m < n$,

$$|x_n - x_m| \leq \sum_{i=m}^{n-1} |x_{i+1} - x_i| \leq c \sum_{i=m}^{n-1} r^i = \frac{cr^m(1 - r^{n-m})}{1 - r}.$$

When m, n are very large, this last expression is small since $0 < r < 1$ (use logarithms), so it can be seen that $\{x_n\}$ is Cauchy. ○–●

▶ $[\mathbf{3.37}]$: Answer the (Why?) in the above example with an appropriate induction proof.

●–○ For $n = 1$, it is clear. Assume the claim for $n - 1$. Then

$$s_{2^n} = s_{2^{n-1}} + \frac{1}{2^{n-1} + 1} + \cdots + \frac{1}{2^n} \geq s_{2^{n-1}} + \frac{2^{n-1}}{2^n} \geq \frac{n}{2} + \frac{1}{2}.$$

So the claim follows by induction. ○–●

▶ $[\mathbf{3.38}]$: Prove the previous theorem.

●–○ The partial sums of $\sum(x_j \pm y_j)$ are the partial sums of $\sum x_j$ plus or minus the partial sums of $\sum y_j$. See Proposition 2.3. The strategy is the same for $\sum cx_j$. ○–●

▶ $[\mathbf{3.41}]$: Suppose $|\,x_j\,| \geq a\,r^j$, where $a > 0$ and $r > 1$. Show that $\sum_{j=1}^{\infty} x_j$ diverges.

●–○ $\{x_j\}$ can't tend to zero since $|x_j|$ is unbounded. ○–●

▶ $[\mathbf{3.42}]$: For $\{x_j\} \subset \mathbb{X}$ such that $\sum_{j=1}^{\infty} |x_j|$ converges, show that

$$\left|\,|x_1| - \left|\sum_{j=2}^{\infty} x_j\right|\,\right| \leq \left|\sum_{j=1}^{\infty} x_j\right| \leq \sum_{j=1}^{\infty} |x_j|.$$

This is the natural extension of the triangle and reverse triangle inequalities to infinite series.

●–○ For partial sums this really is the triangle inequality. Indeed:

$$\left|\,|x_1| - \left|\sum_{j=2}^{n} x_j\right|\,\right| \leq \left|x_1 + \sum_{j=2}^{n} x_j\right| = \left|\sum_{j=1}^{n} x_j\right| \leq \sum_{j=1}^{n} |x_j|\,.$$

Now let $n \to \infty$ and use the order properties of limits to finish the solution. ○–●

▶ $[\mathbf{3.43}]$: Show that, if $|r| < 1$, the corresponding geometric series converges to the sum $\frac{1}{1-r}$. To get started, show that $s_n = \frac{1-r^{n+1}}{1-r}$.

•–∘ A simple inductive argument shows that the formula for s_n is true. From that, and from the fact that $\lim r^{n+1} = 0$ for $|r| < 1$ (which you may prove using logarithms) the assertion follows. ∘–•

▶ $[\mathbf{3.45}]$: Complete the proof of the above theorem by showing that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow |s_n - s| < \epsilon$. (Hint: The value of n must be either even or odd.)

•–∘ Set $s \equiv s_{odd} = s_{even}$. By the proof, it is possible to choose N_1 so that $2n > N_1$ implies $|s_{2n} - s| < \epsilon$. Choose N_2 so that $2n + 1 > N_2$ implies $|s_{2n+1} - s| < \epsilon$. That we can do this is easily seen, as it is just the definition of a limit. Any n greater than $N \equiv \max(N_1, N_2)$ is either even or odd so satisfies $|s_n - s| < \epsilon$. ∘–•

▶ $[\mathbf{3.47}]$: Show that the condition "$0 \leq x_j \leq y_j$ for $j = 1, 2, \ldots$" in The Comparison Theorem can be weakened to "$0 \leq x_j \leq y_j$ for $j \geq N$ for some $N \in \mathbb{N}$" and the same conclusion still holds.

•–∘ Changing the value of a finite number of the terms of a series doesn't affect whether or not it converges, by, e.g., the Cauchy condition. ∘–•

▶ $[\mathbf{3.48}]$: Prove that $s_M \leq t_n$ for $2^n \leq M < 2^{n+1}$, and that $t_n \leq 2s_{2^n}$ for $n \geq 1$.

•–∘ We prove the first claim by complete induction on M. Suppose the first claim true for $1, \ldots, M-1$. (It is true for $M = 1$.) Then $s_{2^n-1} \leq t_{n-1}$ by the inductive hypothesis. So

$$s_M = s_{2^n-1} + x_{2^n} + x_{2^n+1} + \cdots + x_M,$$

and

$$t_n = t_{n-1} + 2^{n-1}x_{2^{n-1}}.$$

By the inductive hypothesis, in the first terms on the right of the two equations, we have $s_{2^n-1} \leq t_{n-1}$. It is also true that, since $\{x_n\}$ is nonincreasing,

$$x_{2^n} + x_{2^n+1} + \cdots + x_M \leq 2^{n-1}x_{2^n} \leq 2^{n-1}x_{2^{n-1}}$$

which establishes the inductive step. Similarly, the result on t_n can be proved by induction, because

$$t_n - t_{n-1} = 2^{n-1}x_{2^{n-1}} \leq x_{2^{n-1}+1} + x_{2^{n-1}+2} + \cdots + x_{2^n} \leq s_{2^n} - s_{2^{n-1}}.$$

∘–•

▶ $[\mathbf{3.49}]$: Show that a given p-series is convergent if and only if $p > 1$.

•–∘ Apply the Cauchy condensation test, and note that

$$\sum_{j=0}^{\infty} 2^{j(1-p)}$$

converges if and only if $p > 1$ (by the geometric series). ∘–•

▶ $[\mathbf{3.50}]$: Suppose $\sum_{j=1}^{\infty} \mathbf{x}_j$ and $\sum_{j=1}^{\infty} \mathbf{y}_j$ are each series of non-zero terms in $\mathbb{R}^k$ such that

$$\lim \frac{|\,\mathbf{x}_j\,|}{|\,\mathbf{y}_j\,|} = L > 0.$$

What conclusion can you make? What if $\sum_{j=1}^{\infty} z_j$ and $\sum_{j=1}^{\infty} w_j$ are each series of non-zero terms in $\mathbb{C}$ such that

$$\lim \frac{|\,z_j\,|}{|\,w_j\,|} = L > 0.$$

What conclusion can you make?

●–○ $\mathbf{x}_j$ converges *absolutely* iff $\mathbf{y}_j$ does. To see this, apply the Limit Comparison Test to $|\mathbf{x}_j|$ and $|\mathbf{y}_j|$. WARNING: One sequence out of $\{\mathbf{x}_j\}, \{\mathbf{y}_j\}$ may converge without the other doing so since the sequences are not nonnegative (indeed, are not even real). Consider $(\frac{1}{n}, 0)$ and $((-1)^n \frac{1}{n}, 0)$ for instance.

The situation for $\mathbb{C}$ is similar. ○–●

▶ $[\mathbf{3.51}]$: Prove part $b)$ of the above proposition. To get started, consider that $a_j^+ - a_j^- = a_j$, and $a_j^+ + a_j^- = |a_j|$.

●–○ Note that $\left|a_j^+\right| \leq |a_j|$ and apply the comparison test. Similarly for a_j^-. ○–●

▶ $[\mathbf{3.52}]$: Prove the above proposition.

●–○ If, say, $\sum a_j^+ < \infty$, then $\sum a_j^+$ converges, and so does $\sum a_j^- = \sum (a_j^+ - a_j)$, and so does

$$\sum |a_j| = \sum (a_j^+ + a_j^-).$$

So we get absolute convergence, a contradiction. ○–●

▶ $[\mathbf{3.53}]$: Answer the (Why?) question in the above proof.

●–○ $\lim a_k^- = 0$ because $\lim a_k = 0$, since $\sum a_k$ converges conditionally. Now take any subsequence of a_k^-; it must also tend to zero. ○–●

▶ $[\mathbf{3.54}]$: Answer the two (Why?) questions in the above proof, and prove the above theorem in the case where $a_j < 0$ for at least one j.

●–○ The first one follows since T_n is a sum of various a_j (possibly oout of order), while S is the (infinite) sum of all the a_j. The latter follows by reversing the roles of S and T: $\sum a_j$ is a rearrangement of $\sum_j a_j{}'$, and one repeats the above argument with S and T reversed. The case with negative a_j's follows by splitting the sequence into positive and negative parts. ○–●

▶ $[\mathbf{3.55}]$: Answer the four (Why?) questions from the proof of the previous theorem.

●–○
(1) $B_n' = B - B_n$. Since the right hand side tends to zero by definition, $\lim B_n' = 0$.
(2) Each B_j' for $j > N$ is of absolute value less than ϵ.

(3) Note that N and the $B_i', 1 \leq i \leq N$ are fixed, and each $|a_{n-k}| \to 0$ for each $k = 1, 2, \ldots, N$.

(4) The lim inf is always less than or equal to the lim sup but is nonnegative in this case, since the terms are nonnegative.

chapter 4

▶ $[\mathbf{4.1}]$: Show that the real axis lies in the range of f in the previous example, and therefore the range of f is all of $\mathbb{C}$.

For nonnegative real numbers, we can take the (normal, real-valued) square root. For negative real numbers x, we can take $i\sqrt{-x}$ where $\sqrt{-x}$ is the ordinary square root of the positive real number $-x$.

▶ $[\mathbf{4.6}]$: Consider $D^1 \subseteq \mathbb{R}$ and consider $f_j : D^1 \to \mathbb{R}$ for $j = 1, 2, \ldots, p$. Is it true that if f_j is a one-to-one function for $j = 1, 2, \ldots, p$, then the function $\boldsymbol{f} : D^1 \to \mathbb{R}^p$ given by $\boldsymbol{f} = (f_1, f_2, \ldots, f_p)$ is one-to-one? (Yes.) How about the converse? (No.)

If f_1 is one-to-one and $x, y \in D^1$ with $x \neq y$, then $\boldsymbol{f}(x) \neq \boldsymbol{f}(y)$ because the first coordinates differ. For the converse, take $\boldsymbol{f} = (x, x^2)$ with $D^1 = \mathbb{R}, p = 2$; this is one-to-one, but the second coordinate f_2 is not.

▶ $[\mathbf{4.11}]$: Show that the square root function $f : \mathbb{C} \to \mathbb{C}$ given in Definition 3.1 is one-to-one. Is it onto? (No.) What can you conclude about the function $h : \mathbb{C} \to \mathbb{C}$ given by $h(z) = z^2$? Is it f^{-1}? (No.)

It is one-to-one because if $\sqrt{z_1} = \sqrt{z_2}$, then one could square both sides to get $z_1 = z_2$. It is not onto because, for instance, -1 is never returned as a value. Therefore f and h are not inverses.

▶ $[\mathbf{4.14}]$: Prove properties $a), b)$, and $e)$ of the above proposition.

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. Then

$$e^{z_1} e^{z_2} = e^{x_1} e^{iy_1} e^{x_2} e^{iy_2} = e^{x_1} e^{x_2} e^{iy_1} e^{iy_2} = e^{x_1 + x_2} e^{i(y_1 + y_2)} = e^{z_1 + z_2}$$

by Prop. 3.6a) of Chapter 1 (basically, this property for purely imaginary numbers). This proves a). For b), recall that $e^0 = 1$ by the definition, so

$$e^z e^{-z} = e^{z + (-z)} = e^0 = 1.$$

This prove b). b) implies e); zero has no multiplicative inverse.

▶ $[\mathbf{4.22}]$: Establish the following properties of the complex sine and cosine functions. Note that each one is a complex analog to a well-known property of the real sine and cosine functions.

$a)$ $\sin(-z) = -\sin z$ $\quad b)$ $\cos(-z) = \cos z$ $\quad c)$ $\sin^2 z + \cos^2 z = 1$

$d)$ $\sin(2\,z) = 2\,\sin z\,\cos z$ $\quad e)$ $\sin(z + 2\,\pi) = \sin z$ $\quad f)$ $\cos(z + 2\,\pi) = \cos z$

$g)$ $\sin(z + \pi) = -\sin z$ $\quad h)$ $\cos(z + \pi) = -\cos z$ $\quad i)$ $\cos(2\,z) = \cos^2 z - \sin^2 z$

$j)$ $\sin\left(z + \frac{\pi}{2}\right) = \cos z$

•–○ We do a sample. For a), $\sin(-z) = \frac{1}{2i}\left(e^{-iz} - e^{iz}\right) = -\sin z$. For c),

$$\begin{aligned} \sin^2 z + \cos^2 z &= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\ &= \frac{e^{2iz} + e^{-2iz} + 2}{4} + \frac{e^{2iz} + e^{-2iz} - 2}{-4} = 1. \end{aligned}$$

For d),

$$\begin{aligned} 2\sin z \cos z &= 2\frac{1}{4i}(e^{iz} + e^{-iz})(e^{iz} - e^{-iz}) \\ &= \frac{1}{2i}(e^{2iz} - e^{-2iz}). \end{aligned}$$

e) follows from the corresponding periodicity of the exponential function: $e^z = e^{z+2\pi i}$. For g), use the definition and $e^{i\pi} = -1$, so $e^{z+i\pi} = -e^z$. ○–•

▶ $\mathbf{[4.25]}$: Prove the statement at the end of remark 8 above. That is, for a function $f : \mathbb{D} \to \mathbb{Y}$ and x_0 a limit point of $\mathbb{D}$, suppose $\lim_{x\to x_0} f(x) = L$. If $\{x_n\}$ is a sequence in $\mathbb{D}$ such that $x_n \neq x_0$ for all n and $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = L$. What if the sequence does not satisfy the condition that $x_n \neq x_0$ for all n? (In that case, the result fails.)

•–○ Choose $\epsilon > 0$.

There is a $\delta > 0$ such that if $x \in N'_\delta(x_0)$,

$$|f(x) - L| < \epsilon.$$

Also, there exists N such that $n > N$ implies

$$|x_n - x_0| < \delta,$$

and since $x_n \neq x_0$, we have $x_n \in N'_\delta(x_0)$, so we find for $n > N$,

$$|f(x_n) - L| < \epsilon,$$

which proves the claim.

The claim fails if we allow $x_n = x_0$ for some n, e.g., with the function $f(x)$ defined to equal 1 at $x = 0$ and 0 otherwise. Using the sequence $x_n \equiv 0$, we have

$$\lim f(x_n) = \lim 1 = 1,$$

but (prove this!)

$$\lim_{x\to 0} f(x) = 0.$$

○–•

▶ $[\mathbf{4.28}]$: Show that the real function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \exp\big(-1/x^2\big)$ has a limit of 0 as $x \to 0$, but that the complex function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \exp\big(-1/z^2\big)$ does *not* have a limit as $z \to 0$.

•–∘ We first prove the claim about the real function f. Fix $\epsilon > 0$. Choose $X > 0$ via $X = -\ln(\epsilon)$. Then if x is real and $0 < |x| < \delta \equiv \frac{1}{\sqrt{X}}$, we have

$$|e^{-1/x^2}| < |e^{-1/X}| = \epsilon.$$

This proves the claim. For the other half, note that as z approaches 0 along the imaginary axis, the function f takes arbitrarily large values (i.e. note that $f(iy) = e^{1/y^2}$, and as $y \to 0$ this gets arbitrarily large). ∘–•

▶ $[\mathbf{4.29}]$: Answer the (Why?) question in the above proof, and also prove part b).

•–∘ First choose a $\delta_j > 0$ for each j and f_j as in the proof. Then set $\delta \equiv \min \delta_j$. This δ works the same for all the f_j uniformly. ∘–•

▶ $[\mathbf{4.37}]$: Complete the case $A = 0$ from the previous example.

•–∘ Suppose f is nonnegative and $\lim_{x \to x_0} f(x) = 0$. Then given $\epsilon > 0$, we can find $\delta > 0$ so small that $|x - x_0| < \delta$ and $x \in \mathbb{D}, x \neq x_0$ implies

$$|f(x) - 0| < \epsilon^2, \text{ so } |\sqrt{f(x)} - 0| < \epsilon,$$

which proves the assertion in the example, i.e. that

$$\lim_{x \to x_0} \sqrt{f(x)} = 0.$$

∘–•

chapter 5

▶ $[\mathbf{5.1}]$: Prove remark 2 above.

In the case of a limit point, it is clear from the definitions of continuity and limits that f is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$ (and the limit exists). Note that the only reason the hypothesis x_0 is a limit point was used in the remark was because otherwise we have not even defined the limit![1] Now since $\lim_{x \to x_0} x = x_0$ exists, the remark is clear.

▶ $[\mathbf{5.5}]$: Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = az + b$ where $a, b \in \mathbb{C}$. Show that f is continuous on $\mathbb{C}$.

Fix $z_0 \in \mathbb{C}$ and $\epsilon > 0$. Choose $\delta \equiv \frac{\epsilon}{|a|+1}$. Then if $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| = |a(z - z_0)| < \frac{|a|\,\epsilon}{|a| + 1} < \epsilon.$$

Thus f is continuous at z_0. (Note that we added the 1 to $|a|$ to avoid dividing by zero if $a = 0$.)

▶ $[\mathbf{5.7}]$: Show in the above example that it is impossible to choose δ to be independent of x_0. (Hint: For given $\epsilon > 0$, consider $x_0 = \frac{1}{n\,\epsilon}$ and $x = \frac{1}{(n+1)\,\epsilon}$ for sufficiently large $n \in \mathbb{N}$.)

Suppose such a δ existed independent of x_0, corresponding to the tolerance $\epsilon > 0$. This means that whenever $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Consider the consecutive terms of $a_n \equiv \frac{1}{(n+1)\epsilon}$. Since $\lim a_n = 0$, we have for n greater than some fixed M, $|a_n| < \frac{\delta}{2}$. Therefore for $n > M$, we have $|a_n - a_{n+1}| < \delta$. By assumption, $|f(a_n) - f(a_{n+1})| < \epsilon$, but in fact we have equality (indeed, $f(a_n) - f(a_{n+1}) = -\epsilon$) from the definitions.

▶ $[\mathbf{5.8}]$: Consider $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Use this example to show that, as claimed in remark 4 after Definition 1.1, not all continuous functions preserve convergence.

[1] Why didn't we define the limit at a non-limit point? Because it would then be non-unique. Prove this as an exercise.

●─○ Take $\frac{1}{n}$ which converges to 0, though its image under f doesn't converge at all. ○─●

▶ $[\mathbf{5.13}]$: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x$. Show that f is continuous on $\mathbb{R}$.

●─○ For one approach, this is a simple consequence of the identity $\cos x = \sin(x + \frac{\pi}{2})$; the details are left to the reader. Alternatively, the reader could use a similar trigonometric identity as in the text for the cosine (which he or she could derive using $\cos x = \sin(x + \frac{\pi}{2})$, for instance). ○─●

▶ $[\mathbf{5.21}]$: Prove that the real exponential function $\exp : \mathbb{R} \to \mathbb{R}$ given by $\exp(x) = e^x$ is continuous on its domain.

●─○ This exercise is somewhat unusually difficult. Fix $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Note that

$$|e^x - e^{x_0}| = e^{x_0}|e^{x-x_0} - 1|.$$

So, since x_0 is fixed, we see that we are reduced to bounding $|e^{x-x_0} - 1|$. We shall do this in the lemma.

Lemma 1. *When $|y| < 1$, we have*

$$|e^y - 1| \leq 3|y|.$$

Proof of the lemma. We have

$$|e^y - 1| \leq \sum_{n=1}^{\infty} \frac{|y|^n}{n!} \leq |y| \left(\sum_{n=1}^{\infty} \frac{|y|^{n-1}}{n!} \right).$$

But the sum in parenthesis is at most $\sum_{n=0}^{\infty} \frac{1}{n!} = e < 3$ when $|y| < 1$. This proves the lemma. ○─●

Now, return to the original proof. Take $\delta \equiv \min\left(1, \frac{\epsilon}{3e^{x_0}}\right)$. Then, we have

$$|e^x - e^{x_0}| = e^{x_0}|e^{x-x_0} - 1| < \epsilon$$

if $|x - x_0| < \epsilon$, by the lemma. ○─●

▶ $[\mathbf{5.22}]$: Prove the previous proposition.[2]

●─○ The case of x_0 not a limit point is vacuous and uninteresting (all functions from $\mathbb{D}$ are automatically continuous at x_0), so we assume x_0 is a limit point. Recall that the definition of continuity at x_0 is, in this case, that $\lim_{x \to x_0} f(x) = f(x_0)$ (where we require the existence of the limit).

Now all these properties are immediate consequences of the basic properties of limits in Chapter 4. For instance, we prove d): if f, g are continuous at x_0, then

$$\lim_{x \to x_0} (f \cdot g)(x) = (\lim_{x \to x_0} f(x)) \cdot (\lim_{x \to x_0} g(x)) = f(x_0) \cdot g(x_0) = (f \cdot g)(x_0),$$

which implies $f \cdot g$ is continuous at x_0. The proofs of all the other assertions follow this pattern. ○─●

[2] Proposition 1.9

▶ $[\mathbf{5.23}]$: Does property 4 in the above theorem still hold if f has codomain $\mathbb{R}$ while g has codomain $\mathbb{R}^p$?

•–∘ Yes. The proof is left to the reader; it is similar to the solution in Exercise 5.22, but taking into account the corresponding property of limits for dot products. ∘–•

▶ $[\mathbf{5.25}]$: Prove that the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$.

a) $f(x) = c$ for $c \in \mathbb{R}$

b) $f(x) = x$

c) $f(x) = x^n$ for $n \in \mathbb{Z}^+$

d) $f(x) = c_0 + c_1\,x + c_2\,x^2 + \cdots + c_n\,x^n$ for $c_j \in \mathbb{R}$ and $n \in \mathbb{N}$ with $c_n \neq 0$.

This establishes that all real polynomials are continuous on $\mathbb{R}$.

•–∘

(1) For any $\epsilon > 0$, choose $\delta \equiv 1$ in the ϵ-δ definition of continuity.
(2) Choose $\delta \equiv \epsilon$. This can be done regardless of the point x_0 at which one is seeking to establish continuity.
(3) Induction using b) of this exercise and d) of Proposition 1.9. (We know that x is continuous, hence $x^2 = x \cdot x$, hence $x^3 = x \cdot x^2$, and so on...)
(4) Use b) and c) of Theorem 1.9 and c).

∘–•

▶ $[\mathbf{5.34}]$: Show that the function $f : \mathbb{R}^k \to \mathbb{R}$ given by $f(\mathbf{x}) = -|\mathbf{x}|^2$ is continuous on $\mathbb{R}^k$.

•–∘ $f(\mathbf{x}) = -\mathbf{x} \cdot \mathbf{x}$, and use Proposition 1.9. ∘–•

▶ $[\mathbf{5.36}]$: Prove the following corollary to Proposition 1.15: *Suppose $f : \mathbb{D} \to \mathbb{Y}$ is continuous at $x_0 \in \mathbb{D}$ where x_0 is an interior point of $\mathbb{D}$. Then there exists a $\delta > 0$ such that f is bounded on $N_\delta(x_0)$.*

•–∘ There exists a $\delta > 0$ such that f is bounded on $N_\delta(x_0) \cap \mathbb{D}$. Reducing δ, we may assume $N_\delta(x_0) \subset \mathbb{D}$ by assumption about x_0's being an interior point; shrinking the neighborhood will not affect the boundedness of f. Then f is bounded on $N_\delta(x_0) \subset \mathbb{D}$. ∘–•

▶ $[\mathbf{5.37}]$: Answer the (Why?) question in the above proof, and then prove the case of the above theorem where $\mathbb{Y}$ is a proper subset of $\mathbb{R}$, $\mathbb{R}^p$, or $\mathbb{C}$.

•–∘ Let $x \in f^{-1}(U)$. Then $N_\delta(x) \subset V$ by the definition of V as a union of δ-neighborhoods, and x belongs to the former set, which answers the (Why?) question.

If $f : \mathbb{D} \to \mathbb{Y}$ is continuous, we may replace $\mathbb{Y}$ by the ambient space $\mathbb{R}, \mathbb{R}^p$, or $\mathbb{C}$ and still have a continuous function f. This follows from the definition of continuity. If V is open in the ambient space, then $f^{-1}(V)$ is open in $\mathbb{D}$ by the fraction of Theorem 1.16 that was proved. But $f^{-1}(V) = f^{-1}(V \cap \mathbb{Y})$ since the range of f is contained in $\mathbb{Y}$. Thus every set of the form $V \cap \mathbb{Y}$ with V open has an inverse image open in $\mathbb{D}$, proving the theorem by the definition of relative openness (the relatively open sets are precisely of the form $V \cap \mathbb{Y}$ for V open in the ambient space). ○—●

► $[\mathbf{5.38}]$: Note that the condition on U that it be "open in $\mathbb{Y}$" cannot be changed to "open and a subset of $\mathbb{Y}$." Trace through the example of the function $f : \mathbb{R} \to \mathbb{Y}$ given by

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

with $\mathbb{Y} = \{-1, 1\}$ to see why. In this example, the conditions of the altered theorem would be satisfied, but the conclusion obviously doesn't hold for f.

●—○ The only subset of $\mathbb{Y}$ that is open in $\mathbb{R}$ is the empty set, and the inverse image of the empty set (namely, the empty set again), is open. Yet f is not continuous at 0. ○—●

► $[\mathbf{5.40}]$: Show that $f : \mathbb{D} \to \mathbb{Y}$ is continuous on $\mathbb{D}$ if and only if $f^{-1}(B)$ is closed in $\mathbb{D}$ for every B closed in $\mathbb{Y}$.

●—○ Take complements with respect to $\mathbb{Y}$. In detail, f is continuous iff A open in $\mathbb{Y}$ implies $f^{-1}(A)$ open in $\mathbb{D}$. This is equivalent to stipulating that when B is *closed* in $\mathbb{Y}$, we have $f^{-1}(\mathbb{Y} - B)$ open in $\mathbb{D}$, since open sets are precisely the complements of the closed ones.

But $f^{-1}(\mathbb{Y} - B) = \mathbb{D} - f^{-1}(B)$. So this in turn is equivalent to stipulating that if B is closed in $\mathbb{Y}$, we have $\mathbb{D} - f^{-1}(B)$ open in $\mathbb{D}$—or equivalently, if B closed in $\mathbb{Y}$, then $f^{-1}(B)$ closed in $\mathbb{D}$. This last condition is then equivalent to continuity, which completes the solution. ○—●

► $[\mathbf{5.42}]$: Can you find an example of a discontinuous function $f : \mathbb{D} \to \mathbb{Y}$ such that for some $A \subset \mathbb{D}$, $f(\overline{A}) \supset \overline{f(A)}$?

●—○ Take $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

and A to be $(0, \infty)$. ○—●

► $[\mathbf{5.43}]$: As claimed in the above proof, verify that $K \subset \bigcup_\alpha V_\alpha$ and that $f(K) \subset \bigcup_{i=1}^r W_{\alpha_i}$.

●—○ If $x \in K$, then $f(x) \in f(K)$, so $f(x) \in W_\alpha$ for some α. This means $x \in f^{-1}(W_\alpha)$. Thus $K \subset \bigcup_\alpha f^{-1}(W_\alpha) \subset \bigcup_\alpha V_\alpha$.

For the second part, take images (via f) of the containment

$$K \subset K \cap (V_{\alpha_1} \cup \cdots \cup V_{\alpha_r}) = f^{-1}(W_{\alpha_1}) \cup \cdots \cup f^{-1}(W_{\alpha_r}).$$

○—●

► $[\mathbf{5.46}]$: Let $f : (0,1) \to \mathbb{R}$ be given by $f(x) = \frac{1}{x} \sin \frac{1}{x}$. Show that f is continuous on $(0,1)$. Is $f\left((0,1)\right)$ compact?

●—○ Continuity is left to the reader. $f((0,1))$ is unbounded—consider the images of $\frac{2}{(2n+1)\pi}$, for instance—so it isn't compact. ○—●

▶ $[\mathbf{5.47}]$: Finish the proof of the above theorem.

•–∘ First, pick $y \in V \neq \emptyset$. There exists $x \in \mathbb{D}$ such that $f(x) = y$, so $x \in f^{-1}(V)$ and $f^{-1}(V) \neq \emptyset$; similarly, $f^{-1}(W)$ is nonempty. In words, V is a nonempty subset of the image of f, so its inverse image must be nonempty too.

Next, if $x \in f^{-1}(V) \cap \overline{f^{-1}(W)}$, then $f(x) \in V$, and $f(x) \in \overline{W}$ by Proposition 1.18. This is a contradiction as $V \cap \overline{W} = \emptyset$. This proves the first intersection is empty, and the second is similar. ∘–•

▶ $[\mathbf{5.51}]$: Prove the general case of the intermediate value theorem. Why is it not as generally applicable as Theorem 1.27?

•–∘ Replace f with $f - c$ if $f(a) < f(b)$, or $c - f$ otherwise, and apply the special case of the theorem already proved (namely, when $f(a) < 0$ and $f(b) > 0$). ∘–•

▶ $[\mathbf{5.56}]$: Prove the above proposition.

•–∘ Suppose f is left and right-continuous at x_0. Fix $\epsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ so that:

$$0 \leq x - x_0 < \delta_1 \quad \text{implies} \quad |f(x) - f(x_0)| < \epsilon.$$

and

$$0 \geq x - x_0 > -\delta_2 \quad \text{implies} \quad |f(x) - f(x_0)| < \epsilon.$$

Take $\delta \equiv \min(\delta_1, \delta_2)$; then if $|x - x_0| < \delta$, we have either $0 \leq x - x_0 < \delta_1$ or $0 \geq x - x_0 > -\delta_2$, and in either case:

$$|f(x) - f(x_0)| < \epsilon.$$

The converse is left to the reader. ∘–•

▶ $[\mathbf{5.57}]$: Let $I = [a, b]$ be an interval of real numbers.

a) Show that any subset $J \subset I$ of the form $J = [a, b]$, $(c, b]$, $[a, c)$, or (c, d) is open in I.

b) Suppose $f : I \to \mathbb{R}$ is one-to-one and continuous on I. Let $m \equiv \min\{f(a), f(b)\}$, and let $M \equiv \max\{f(a), f(b)\}$. Show that $f(I) = [m, M]$, and that f must be either increasing or decreasing on I.

c) Suppose $f : I \to [m, M]$ is one-to-one, onto, and continuous on I. For each of the J intervals described in part a), prove that $f(J)$ is open in $[m, M]$.

•–∘ a) $[a, b] = [a, b] \cap \mathbb{R}$, $(c, b] = [a, b] \cap (c, \infty)$, $[a, c) = [a, b] \cap (-\infty, c)$, $(c, d) = [a, b] \cap (c, d)$. Thus all those four sets are open in I.

b) We will show that f is strictly monotone, and that the minimum and maximum values of f are therefore m, M. Assuming this, let us prove the claim in b). We have $f(I) \subset [m, M]$ by definition. $f(I) \supset [m, M]$ by an application of the Intermediate Value Theorem. (If $f(c) = m$, $f(d) = M$, then between c and d each value in $[m, M]$ is taken.)

Now it is to be shown that f is strictly monotone. Otherwise, since f is one-to-one, there would be $c < d < e \in [a, b]$ such that $f(d) > f(c)$ and $f(d) > f(e)$, or such that $f(d) < f(c)$ and $f(d) < f(e)$. (Why?)

In either case, one can derive a contradiction; take the former to fix ideas. Choose some q such that $q \in (f(c), f(d)) \cap (f(e), f(d))$. Then there exist, by the intermediate value theorem, $x_1 \in (c, d)$ and $x_2 \in (d, e)$ with $f(x_1) = f(x_2) = q$. Clearly $x_1 \neq x_2$. This contradicts the assumption that f is one-to-one.

c) Such an interval is mapped onto an interval of one of those four types (by the intermediate value theorem).

○—●

▶ $[\mathbf{5.67}]$: What if the domain of the function in the previous example is changed to $(-1, \infty)$? Does the domain matter at all?

●—○ No, because f "blows up" as $x \to -1$ from the right. ○—●

▶ $[\mathbf{5.71}]$: Fix $\alpha \in \mathbb{R}$ and consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^\alpha \sin \frac{1}{x} & \text{for } 0 < x \leq 1, \\ 0 & \text{for } x = 0. \end{cases}$$

For what values of α is the function uniformly continuous on $[0, 1]$?

●—○ f is uniformly continuous iff it is continuous. It is not continuous if $\alpha \leq 0$, because if $\alpha < 0$ it is unbounded near zero (take $x_n = \frac{2}{\pi(2n+1)}$, a sequence tending to zero on which f is unbounded) or if $\alpha = 0$, it oscillates wildly (take $x_n = \frac{2}{\pi(2n \pm 1)}$ or graph it).

If $\alpha > 0$ the function is uniformly continuous: it is continuous on $(0, 1]$ clearly, and is continuous at zero since $\lim_{x \to 0} f(x) = 0$, by the squeeze theorem. ○—●

▶ $[\mathbf{5.72}]$: What happens if in the statement of the above theorem, you replace uniform continuity with mere continuity?

●—○ It no longer works. (Try $f(x) = \frac{1}{x}$ and the sequence $\{\frac{1}{n}\}$.) ○—●

▶ $[\mathbf{5.74}]$: Show that $h^{-1}(V) = f^{-1}(V) \cup g^{-1}(V)$ as claimed in the above proof. Also, answer the (Why?) question posed there.

●—○ $x \in h^{-1}(V)$ iff $h(x) \in V$ iff either $x \in A$ and $f(x) \in V$ or $x \in B$ and $g(x) \in V$. This is equivalent to $x \in f^{-1}(V) \cup g^{-1}(V)$.

Answer to the (Why?) question: if $f^{-1}(V)$ is closed in A (as we know from continuity), i.e. $f^{-1}(V) = G \cap A$ for a closed set G, then we know that $f^{-1}(V)$ is a closed set in the ambient space (A being closed). So a fortiori $f^{-1}(V)$ is closed in $\mathbb{D}$. Ditto for $g^{-1}(V)$. ○—●

▶ $[\mathbf{5.75}]$: In the statement of the pasting theorem, suppose A and B are presumed to be *open* sets rather than closed. Prove that the result of the pasting theorem still holds.

⊸ Repeat the proof, using the criterion for continuity: a function is continuous iff the inverse image of an *open* set is open.

⊸

▶ $[\mathbf{5.76}]$: Suppose $A = (-\infty, 1]$, $B = (1, \infty)$, and that $h : \mathbb{R} = A \cup B \to \mathbb{R}$ is given by

$$h(x) = \begin{cases} x^2 & \text{if } x \in A, \\ 2 & \text{if } x \in B. \end{cases}$$

Is h continuous at $x = 1$? What does this exercise illustrate?

⊸ No, h fails to be continuous at $x = 1$; it illustrates that f and g have to agree on $A \cap B$ in the pasting theorem. ⊸

▶ $[\mathbf{5.77}]$: Answer the three (Why?) questions in the proof of the above theorem. Also, note that there were three places where we claimed a sequence existed convergent to x_0, ξ, and η, respectively. Show that the result of the theorem does not depend on the choice of convergent sequence chosen in each case. Finally, establish that the $\widetilde{f}$ constructed in the above proof is unique.

⊸ Question 1: We know that $x_0 \in \mathbb{D}'$, because $x_0 \notin \mathbb{D}$. Thus, for each $n \in \mathbb{N}$ there exists $x_n \in \mathbb{D} \cap N'_{1/n}(x_0)$. This sequence converges to x_0.

Question 2: Cf. Proposition 2.10 (the image of a Cauchy sequence under a uniformly continuous map is Cauchy).

Question 3: The first two hold for all sufficiently large N because $\lim x_n = \xi, \lim x'_n = \eta$. The second two hold for all sufficiently large N because $\tilde{f}(\xi) = \lim f(x_n), \tilde{f}(\eta) = \lim f(x'_n)$.

The assertion about the independence of the convergent sequence follows from the uniqueness, which we prove: if $\xi \in \bar{\mathbb{D}} - \bar{D}$ pick *any* sequence in $\mathbb{D}$, $x_n \to \xi$. If $\tilde{f}$ is *any* continuous extension of f, then (by continuity) $\tilde{f}(\xi) = \lim \tilde{f}(x_n) = \lim f(x_n)$.

Thus there can be at most one continuous extension of f—the value at ξ is determined by its values on $\{x_n\} \subset \mathbb{D}$, and $\xi \in \bar{\mathbb{D}}$ was arbitrary. ⊸

▶ $[\mathbf{5.78}]$: Prove the above proposition.

⊸ One implication is already known (that uniformly continuous functions can be extended to their closures). The other follows easily from the fact that $\mathbb{D}$ is compact (the Heine-Borel Theorem) and continuous functions on a compact set are uniformly continuous. If f extends, the extension must be uniformly continuous on $\overline{\mathbb{D}}$, hence on $\mathbb{D}$. ⊸

▶ $[\mathbf{5.81}]$: Consider the sequence of complex functions $f_n : N_1(0) \subset \mathbb{C} \to \mathbb{C}$ described by $f_n(z) = z^n$ for $n \in \mathbb{N}$, and the complex function $f : N_1(0) \subset \mathbb{C} \to \mathbb{C}$ given by $f(z) \equiv 0$ on $N_1(0) \subset \mathbb{C}$. What can you say about $\lim_{n\to\infty} f_n(z)$ on $N_1(0)$?

⊸ $\lim_{n\to\infty} f_n(z) = f(z)$. This is simply because if $z \in N_1(0)$ is fixed, $\lim z^n = 0$ (since $|z^n| = |z|^n \to 0$). ⊸

▶ $[\mathbf{5.85}]$: For $D = \{z \in \mathbb{C} : Re(z) \neq 0\}$, consider the sequence of functions $f_n : D \to \mathbb{C}$ given by $f_n(z) = e^{-nz}$ for $n \in \mathbb{N}$. Find $\lim_{n\to\infty} f_n(z)$.

●-○ The limit function is zero if $Re(z) > 0$, because $|e^{-nz}| = e^{-nRe(z)}$ and this last sequence clearly tends to zero. The limit doesn't exist if $Re(z) < 0$, because for such z, $f_n(z)$ is unbounded, by a similar calculation of the absolute values. ○-●

▶ $[\mathbf{5.86}]$: In a previous exercise, we saw that $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = x^n$ for $n \in \mathbb{N}$ converged to $f(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$. Is the convergence uniform?

●-○ No: the limit function isn't continuous. ○-●

▶ $[\mathbf{5.87}]$: Let $f_n : (0,1] \to \mathbb{R}$ be given by $f_n(x) = \begin{cases} 1 & \text{for } 0 < x \le \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} < x \le 1. \end{cases}$ Note that the functions $\{f_n\}$ are *not* continuous on $(0,1]$. Show that the limit function $f(x) \equiv \lim f_n(x)$ *is* continuous on $(0,1]$, and the convergence is *not* uniform.

●-○ The limit function is identically zero (indeed, if $x \in (0,1]$ is fixed, $f_n(x) = 0$ for all large enough n) so is continuous. If the convergence were uniform, there would exist N so that $n > N$ implied $|f_n(x)| < \frac{1}{2}$ for all $x \in (0,1]$. This is not true if $x < \frac{1}{n}$. ○-●

▶ $[\mathbf{5.89}]$: Suppose in the statement of the previous theorem that $\mathbb{Y}$ is presumed to be a subset of $\mathbb{R}$, $\mathbb{R}^p$, or $\mathbb{C}$. In this case, what else must be presumed about $\mathbb{Y}$ in order for the argument given in the above proof to remain valid?

●-○ $\mathbb{Y}$ must be closed in order that the limit function take its values in $\mathbb{Y}$. (If $\mathbb{Y}$ is closed, then limits of convergent or Cauchy sequences in $\mathbb{Y}$ belong to $\mathbb{Y}$, so the proof goes through with no changes.) ○-●

▶ $[\mathbf{5.98}]$: Prove the above theorem.

●-○ Cf. Theorem 3.8—it implies this result, when applied to the sequence of partial sums. ○-●

▶ $[\mathbf{5.100}]$: Consider $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ where $z \in \mathbb{C}$.

a) For what values of z, if any, does the series converge?

b) Is the convergence uniform on $\mathbb{C}$?

c) What if z is restricted to $\{z \in \mathbb{C} : |z| \le r\}$, for some $r > 0$?

●-○ It converges for all z by the M-Test. The series

$$\sum_{j=0}^{\infty} \frac{z^n}{n!}$$

is dominated by, for $|z| \le M$,

$$\sum_{j=0}^{\infty} \frac{M^n}{n!},$$

which converges (to e^M) as we already know. Thus the original series converges uniformly on $N_M(0)$ for all M, and thus it converges for each $z \in \mathbb{C}$.

The convergence isn't uniform on $\mathbb{C}$: if it were, then the partial sums would be uniformly Cauchy on $\mathbb{C}$, so there would exist a very large n such that $|s_{n+1}(z) - s_n(z)| < 1$, where $s_n(z) = \sum_{j=0}^{n} \frac{z^j}{j!}$ are the partial sums. But this difference, $s_{n+1}(z) - s_n(z)$, is just $\frac{z^{n+1}}{(n+1)!}$. This difference is *unbounded* on $\mathbb{C}$ and thus can't be less than 1 everywhere.

We have answered a), b), and also c) which we proved alongside a) at the beginning (yes, by the M-Test). ○—●

▶ $[\mathbf{5.101}]$: Establish the following properties of $d_{\mathbb{D}}$ defined above.

a. Show that $|d_{\mathbb{D}}(x) - d_{\mathbb{D}}(y)| \leq |x - y|$ for all $x, y \in \mathbb{X}$, and therefore $d_{\mathbb{D}}$ is uniformly continuous on $\mathbb{X}$.

b. Show that $d_{\mathbb{D}}(x) \geq 0$ for all $x \in \mathbb{X}$, and that $d_{\mathbb{D}}(x) = 0$ if and only if $x \in \bar{\mathbb{D}}$.

●—○ For a), fix $x, y \in \mathbb{X}$, pick $\epsilon > 0$, and pick $t \in \mathbb{D}$ such that $|x - t| < d_{\mathbb{D}}(x) + \epsilon$. (This is possible by the basic properties of infs.) Now by the triangle inequality, $|y - t| \leq |x - t| + |y - x| < |x - y| + d_{\mathbb{D}}(x) + \epsilon$. Letting $\epsilon \to 0$ gives $|y - t| \leq |x - y| + d_{\mathbb{D}}(x)$, which means that $d_{\mathbb{D}}(y) \leq |x - y| + d_{\mathbb{D}}(x)$. (This is because $t \in \mathbb{D}$ and $d_{\mathbb{D}}(y)$ involves taking the inf.) Subtracting yields $d_{\mathbb{D}}(y) - d_{\mathbb{D}}(x) \leq |x - y|$. Interchanging x and y gives $d_{\mathbb{D}}(x) - d_{\mathbb{D}}(y) \leq |x - y|$. Combining the two, we have:

$$|d_{\mathbb{D}}(x) - d_{\mathbb{D}}(y)| \leq |x - y|\,,$$

which proves a).

For b), the values $|x - a|$ for $a \in \mathbb{D}$ are nonnegative, so their inf, which is $d_{\mathbb{D}}(x)$, is nonnegative. Next, fix x. Then, $d_{\mathbb{D}}(x) = 0$ iff for all $r > 0$ there exists $a \in D$ such that $|x - a| < r$ iff for all $r > 0$ there exists $a \in D$ with $x \in N_r(a)$ iff $x \in \bar{\mathbb{D}}$. ○—●

▶ $[\mathbf{5.102}]$: Show that $\mathbb{D}^+$ and $\mathbb{D}^-$ are closed in $\mathbb{D}$, and that $\mathbb{D}^- \cap \mathbb{D}^+ = \varnothing$. This is sufficient to show that $d_{\mathbb{D}^-}(x) + d_{\mathbb{D}^+}(x)$ is never 0, and therefore G is well defined.

●—○ The sets $\mathbb{D}^+, \mathbb{D}^-$ are the inverse images under g of closed sets in $\mathbb{R}$, hence closed. They are the inverse images under g of disjoint intervals, hence are disjoint.

Note that if $d_{\mathbb{D}^-}(x) + d_{\mathbb{D}^+}(x)$, then by the properties of distance functions we would have $x \in \mathbb{D}^- \cap \mathbb{D}^+$, a contradiction. ○—●

▶ $[\mathbf{5.103}]$: Prove properties $(i), (ii), (iii)$ listed in the above proof.

●—○ These all follow because $d_{\mathbb{D}^-}$ is zero on $\mathbb{D}^-$ and $d_{\mathbb{D}^+}$ is zero on $\mathbb{D}^+$, since these distance functions are alwayas nonnegative. ○—●

▶ $[\mathbf{5.104}]$: Verify that $F(x) = f(x)$ for all $x \in \mathbb{D}$, and that $|F(x)| \leq M$ on $\mathbb{D}$.

●—○ The first equality is because the the F_j were so arranged that that the differences between $\sum_{j=1}^{N} F_j(x)$ and f tend to zero on $\mathbb{D}$. The bound $|F(x)| \leq M$ follows because $|F_j(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{j-1} M$, and we can use the geometric series to bound it. ○—●

chapter 6

▶ $[\mathbf{6.1}]$: In the above discussion, we started with the ϵ, δ-version of the definition and derived the linear approximation version from it. Beginning with the linear approximation version, derive the ϵ, δ-version, thus establishing the equivalence of these two versions of the derivative definition, and hence completing the proof of Theorem 1.2.

Straight from the definition: if the linear approximation version holds, then

$$\lim_{x \to a} \frac{f(x) - f(a) - A(x-a)}{x-a} = 0;$$

therefore given $\epsilon > 0$, there exists a $\delta > 0$ such that $|x-a| < \delta$ implies

$$\left| \frac{f(x) - f(a) - A(x-a)}{x-a} \right| < \epsilon.$$

This in turn means

$$|f(x) - f(a) - A(x-a)| \leq \epsilon \, |x-a| \, ,$$

for $0 < |x-a| < \delta$ (actually we can put a less than sign but we do not need to). The inequality trivially holds for $x = a$, and we have proved that the ϵ, δ version holds.

▶ $[\mathbf{6.3}]$: Suppose that $\lim_{x \to a} \frac{f(x)-f(a)-A\,(x-a)}{|x-a|}$ exists and equals L. Is it necessarily true that $\lim_{x \to a} \frac{f(x)-f(a)-A\,(x-a)}{x-a}$ exists as well? And if so, is it necessarily equal to L?

No. Try $f(x) = |x|$, $a = 0$, and $A = 0$.

▶ $[\mathbf{6.4}]$: Now suppose that $\lim_{x \to a} \frac{f(x)-f(a)-A\,(x-a)}{x-a}$ exists and equals L. Is it necessarily true that $\lim_{x \to a} \frac{f(x)-f(a)-A\,(x-a)}{|x-a|}$ exists as well? And if so, is it necessarily equal to L?

No. Try $f(x) = x$, $a = 0$, and $A = 0$.

▶ $[\mathbf{6.7}]$: For each $f(x)$ below, use the difference quotient formulation to verify that the given $f'(a)$ is as claimed. Then use the ϵ, δ-version to prove that the derivative is the given $f'(a)$.

$a)$	$f(x) = c$ for $D^1 = \mathbb{R},$ constant $c \in \mathbb{R},$	$f'(a) = 0$ for all $a \in D^1$
$b)$	$f(x) = x^3$ for $D^1 = \mathbb{R},$	$f'(a) = 3a^2$ for all $a \in D^1$
$c)$	$f(x) = x^{-1}$ for $D^1 = \mathbb{R} \setminus \{0\},$	$f'(a) = -a^{-2}$ for all $a \in D^1$
$d)$	$f(x) = x^n$ for $D^1 = \mathbb{R},\ n \in \mathbb{Z}^+,$	$f'(a) = na^{n-1}$ for all $a \in D^1$
$e)$	$f(x) = x^n$ for $D^1 = \mathbb{R} \setminus \{0\},\ n \in \mathbb{Z}^-,$	$f'(a) = na^{n-1}$ for all $a \in D^1$
$f)$	$f(x) = \sin x$ for $D^1 = \mathbb{R},$	$f'(a) = \cos a,$ for all $a \in D^1$
$g)$	$f(x) = \cos x$ for $D^1 = \mathbb{R},$	$f'(a) = -\sin a,$ for all $a \in D^1$

A hint for part f) is to consider the trigonometric identity

$$\sin\theta - \sin\phi = 2\,\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right).$$

•—○ a) The difference quotient version requires no comment. For the ϵ, δ version, choose $\delta = 1$ for any ϵ.

b) This one is less trivial. Note that

$$\lim_{x\to a}\frac{x^3-a^3}{x-a} = \lim_{x\to a} x^2 + ax + a^2 = 3a^2.$$

For the ϵ, δ version note that

$$\left|x^3 - a^3 - 3a^2(x-a)\right| = \left|(x-a)(x^2+ax+a^2-3a^2)\right|.$$

If we choose δ small and $|x-a| < \delta$, the factor on the right will be very small, and we get an expression $\le \epsilon\,|x-a|$.

c) We give only the difference quotient version, leaving the latter to the reader.

$$\frac{x^{-1}-a^{-1}}{x-a} = \frac{a-x}{ax(x-a)} \to -\frac{1}{a^2} \quad \text{as } x \to a.$$

d) Similar to b)

e) Left to the reader

f) By the trigonometric identity, we have

$$\frac{\sin x - \sin a}{x-a} = \cos\left(\frac{x+a}{2}\right)2\sin\left(\frac{x-a}{2}\right)\frac{1}{x-a},$$

and the identity is clear when one recalls that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ (cf. Chapter 4).

g) Cf. part f).

○—•

▶ $[\mathbf{6.9}]$: Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$. Show that f is not differentiable at $a = 0$. Let $g : [0, \infty) \to \mathbb{R}$ be given by $g(x) = |x|$. Show that g *is* differentiable at $a = 0$.

•—○ For the first part, show that the left- and right- hand limits of the difference quotients are different; for the second, note that there is only one way x can approach zero—from the right. (That is, $\lim_{x\to 0^+}\frac{|x|}{x}$ does exist!) ○—•

► $[\mathbf{6.12}]$: Complete the above proof for parts (a), (c), and (d). Also, determine the domains of the functions defined in (a) through (d).

•–∘ (a) and (c) follow easily from the corresponding properties of limits, cf. Chapter 4. (d) can be proved most easily using the Chain Rule below but here is how a direct proof works. The difference quotients are easily transformed into the form

$$\frac{f(x)g(a)-f(a)g(x)}{g(x)g(a)(x-a)} = \frac{1}{g(x)g(a)}\frac{(f(x)g(a)-f(a)g(a))+(f(a)g(a)-f(a)g(x))}{x-a},$$

in which $g(x)\to g(a)$ since g is continuous (cf. Proposition 1.7). In the numerator, we factor $(f(x)g(a)-f(a)g(a)) = g(a)(f(x)-f(a))$, and $(f(a)g(a)-f(a)g(x)) = f(a)(g(a)-g(x))$. Then the definition gives the result. ∘–•

► $[\mathbf{6.13}]$: Prove that the following functions $f:\mathbb{R}\to\mathbb{R}$ are differentiable on $\mathbb{R}$.

a) $f(x)=x$

b) $f(x)=x^n$ for $n\in\mathbb{Z}^+$

c) $f(x)=c_0+c_1\,x+c_2\,x^2+\cdots+c_n\,x^n$ for $c_j\in\mathbb{R}$ and $n\in\mathbb{N}$ with $c_n\neq 0$.

•–∘ a) is easy to see as the difference quotients are all 1. b) follows from the product rule (part (b) of the theorem) by induction starting with a) of this exercise. c) follows from the linearity properties ((a) and (c) in the theorem) combined witb b) of this exercise. ∘–•

► $[\mathbf{6.14}]$: For each of the following, verify that the given $f'(a)$ is as claimed by using an induction proof.

a) $f(x)=x^n$ for $D^1=\mathbb{R},\ n\in\mathbb{Z}^+,\qquad f'(a)=na^{n-1}$ for all $a\in D^1$

b) $f(x)=x^n$ for $D^1=\mathbb{R}\setminus\{0\},\ n\in\mathbb{Z}^-,\quad f'(a)=na^{n-1}$ for all $a\in D^1$

•–∘ Only a) will be verified. Assume it true for $n-1$. We have:

$$\begin{aligned}\frac{d}{dx}x^n &= \frac{d}{dx}(xx^{n-1})\\ &= x\frac{d}{dx}x^{n-1}+x^{n-1}\frac{d}{dx}x \quad \text{(product rule)}\\ &= x(n-1)x^{n-2}+x^{n-1} \quad \text{(by the induction hypothesis)}\\ &= nx^{n-1}.\end{aligned}$$

Since the claim is clear for $n=1$, the solution is complete. ∘–•

► $[\mathbf{6.17}]$: Consider the function $f:D^1\to\mathbb{R}$ given by $f(x)=x^{p/q}$, with $p/q\in\mathbb{Q}$.

a) Find the largest domain D^1 for this function. (Hint: The answer depends on p and q.)

b) For each distinct case determined in part *a)* above, find the values $a\in D^1$ where f is differentiable, and find the derivative.

•–∘ Wlog,[1] p and q have no common factor. If q is odd, we can take the domain to be $\mathbb{R}$ or $\mathbb{R} - \{0\}$, depending upon whether $\frac{p}{q} \geq 0$ or not; this is because q-th roots always exist in $\mathbb{R}$. If q is even, we can define the function as $(x^p)^{1/q}$ and it is thus defined as a real function for $x \in \mathbb{R}^+$ and for 0 according as $p/q \geq 0$ or not.

Now $f^q(x) = x^p$. Differentiate both sides with the Chain Rule, and one obtains:

$$qf^{q-1}(x)f'(x) = px^{p-1},$$

whence

$$f'(x) = \frac{p}{q}x^{p-1-\frac{p}{q}(q-1)} = \frac{p}{q}x^{p/q-1}.$$

This proof isn't valid at $x = 0$, and in fact the function f is differentiable at 0 iff $\frac{p}{q} \geq 1$. This is left to the reader, and is simply a matter of looking at the difference quotients.

The reader should note that the proof was valid for all $x \neq 0$, even negative x, in the appropriate domain (if q was odd). ∘–•

▶ **[6.18]** : Answer the two (Why?) questions in the proof given above, and complete the proof to the theorem by assuming $f(a)$ to be a local maximum with $f'(a) > 0$. Show that a contradiction results. Complete the similar argument for $f(a)$ presumed as a local minimum with $f'(a) \neq 0$.

•–∘ The first (Why?) question is answered by looking at the definition of a derivative, and noting that $f'(a) < 0$.

The second (Why?) question is answered because the quantity in absolute value (namely, $\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right|$, is assumed to be smaller than $-\frac{1}{2}f'(a)$). Consequently we have

$$\frac{f(x) - f(a)}{x - a} \leq \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| + f'(a) < 0.$$

The case of $f'(a) > 0$ is a repetition of the same proof, *mutatis mutandis*, and we sketch it below. Suppose f has a local maximum at a and $f'(a) > 0$. Then we can find $\delta > 0$ such that $0 < |x - a| < \delta$ implies

$$\left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \frac{1}{2}f'(a),$$

and, in view of the reverse triangle inequality, this implies that the difference quotients are positive, so that

$$\frac{f(x) - f(a)}{x - a} > 0$$

for $0 < |x - a| < \delta$. If $a < x < a + \delta$, then necessarily $f(x) > f(a)$. This contradicts the assumption that a was a local maximum.

The case of a local minimum is reduced to the proven case by considering $-f$. ∘–•

[1] An abbreviation for "Without loss of generality."

▶ $[\mathbf{6.21}]$: Answer the (Why?) question in the above proof.

•–○ Otherwise $x_m = c$ and $x_M = d$ (or the reverse of that), which means, since $f(c) = f(d) = 0$, that the maximum and minimum values of f are both zero. In this case, f is constant and $m = M$, in which case the theorem is trivial. ○–•

▶ $[\mathbf{6.23}]$: Prove the following result, known as *the Cauchy mean value theorem.* Let $f, g : [a, b] \to \mathbb{R}$ both be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$[f(b) - f(a)]\ g'(c) = [g(b) - g(a)]\ f'(c).$$

•–○ Apply the mean value theorem to

$$h(x) \equiv g(x)\,(f(a) - f(b)) - f(x)\,(g(a) - g(b))\,,$$

which takes the same value zero at a and b. In particular, there is $c \in (a, b)$ with $h'(c) = 0$, and a little rearrangement shows that this establishes the result. ○–•

▶ $[\mathbf{6.30}]$: Answer the above (Why?) question. Then, suppose in our definition for the derivative of a function $f : D^k \to \mathbb{R}$ we replace $\boldsymbol{A}\,(\mathbf{x} - \mathbf{a})$ with $A\,|\mathbf{x} - \mathbf{a}|$ for some real number A. What is wrong with this proposal? (Hint: Does this definition reduce to the correct derivative value in the special case $k = 1$?)

•–○ It needs to be a real number in order that the limit be defined (since f is a real function). You cannot replace $\boldsymbol{A}(\mathbf{x} - \mathbf{a})$ by $A\,|\mathbf{x} - \mathbf{a}|$ because that would distort the definition—the point is to approximate f by a *linear* function and $A\,|\mathbf{x} - \mathbf{a}|$ isn't linear. ○–•

▶ $[\mathbf{6.31}]$: Verify that the above ϵ, δ-version of Definition 2.1 is in fact equivalent to it.

•–○ Essentially an algebraic multiplication, multiplying both sides by $|\mathbf{x} - \mathbf{a}|$. Cf. section 1 for the proof where $k = 1$, which is no different. ○–•

▶ $[\mathbf{6.33}]$: Prove the above result.

•–○ If $\boldsymbol{A}$ and $\boldsymbol{B}$ were derivatives of f at $\mathbf{a}$, then we would have by algebraic manipulation:

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\boldsymbol{A}(\mathbf{x} - \mathbf{a}) - \boldsymbol{B}(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = 0.$$

Cf. the proof of Proposition 1.3; one simply takes differences. We can rewrite this as

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{(\boldsymbol{B} - \boldsymbol{A})(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = 0.$$

Now choose $\mathbf{x} = \mathbf{a} + h\mathbf{e_i}$ for each i, where h is small, and $\mathbf{e_i}$ are the unit vectors in $\mathbb{R}^k$. We then have:

$$\frac{(\boldsymbol{B} - \boldsymbol{A})h\mathbf{e_i}}{h} \to 0 \text{ as } h \to 0,$$

so since that expression is independent of h,

$$(\boldsymbol{B} - \boldsymbol{A})\mathbf{e_i} = 0$$

for all i. It follows that $\boldsymbol{B} = \boldsymbol{A}$ since they are equal on a basis for $\mathbb{R}^k$.[2] ○—●

▶ $[\mathbf{6.34}]$: In Example 2.3, consider the inequality (6.18). Complete the proof differently by forcing each summand in the parentheses on the left hand side to be less than $\frac{\epsilon}{2}$.

●—○ Take $\delta \equiv \min\left(\sqrt{\frac{\epsilon}{2}}, \frac{\epsilon}{8}\right)$. The details are left to the reader. ○—●

▶ $[\mathbf{6.36}]$: Suppose $f : D^2 \to \mathbb{R}$ is differentiable on all of D^2, where $D^2 = (a,b) \times (c,d)$. Fix $\xi \in (a,b)$, and $\eta \in (c,d)$, and define the functions u and v by

$$u : (a,b) \to \mathbb{R}, \text{ where } u(x) \equiv f(x,\eta)$$

$$v : (c,d) \to \mathbb{R}, \text{ where } v(y) \equiv f(\xi,y).$$

Are the functions u and v differentiable on their domains? If so, find their derivatives.

●—○ The derivatives are the partial derivatives, i.e. $u'(x) = \frac{\partial f}{\partial x}(x,\eta)$ and similarly for v'. See the definition. ○—●

▶ $[\mathbf{6.39}]$: Can you see why the condition that D^k be connected is necessary in Theorem 2.6?

●—○ Take $k = 1$, $D^1 = (0,1) \cup (1,2)$, $f \equiv 1$ on $(0,1)$ and $f \equiv 0$ on $(1,2)$. ○—●

▶ $[\mathbf{6.49}]$: As you might expect, there is a difference quotient definition for the directional derivative which is very similar to Definition 2.9. Write this limit expression and verify that it is consistent with Definition 2.9.

●—○

$$\boldsymbol{f}'_{\mathbf{u}}(\mathbf{a}) = \lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}.$$

This limit might exist even if f isn't differentiable. But if it is differentiable, then recall that the derivative is $\boldsymbol{\nabla f}(\mathbf{x})$, so that

$$\lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a}) - \boldsymbol{f}'(\mathbf{a})(h\mathbf{u})}{|h|} = 0.$$

It is easy to see that this implies

$$\lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a}) - \boldsymbol{f}'(\mathbf{a})(h\mathbf{u})}{h} = 0,$$

and that from this follows

$$\lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h} = \boldsymbol{f}'(\mathbf{a})\mathbf{u} = \boldsymbol{f}'_{\mathbf{u}}(\mathbf{a}).$$

○—●

[2] Alternatively, note that $A_i = \boldsymbol{A}(\mathbf{e_i}) = \boldsymbol{A}(\mathbf{e_i}) = B_i$ for each i.

▶ **[6.45]** : How would you generalize the statement of the above theorem to handle the more general case $f : D^k \to \mathbb{R}$? How would you prove it? What about higher-order mixed partial derivatives? For example, under the right circumstances can we conclude that $f_{yxx} = f_{xyx} = f_{xxy}$?

•–∘ Suppose all the partial derivatives of $f : D^k \to \mathbb{R}$ up to order n exist and are continuous. and suppose $i_1, i_2, \dots, i_n$ is a sequence of elements (not necessarily distinct!) of $\{1, 2, \dots, k\}$. Let σ be a permutation of $\{1, 2, \dots, n\}$. (A permutation is a one-to-one and onto map of $\{1, 2, \dots, n\}$ to itself.) Then:

$$\frac{\partial^n f}{\partial x_{i_n} x_{i_{n-1}} \dots x_{i_1}} = \frac{\partial^n f}{\partial x_{i_{\sigma(n)}} x_{i_{\sigma(n-1)}} \dots x_{i_{\sigma(1)}}}.$$

Now, the above theorem says that the operators $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ *commute* under appropriate conditions. Then

$$\frac{\partial^n}{\partial x_{i_n} x_{i_{n-1}} \dots x_{i_1}}$$

is one product of such operators, while

$$\frac{\partial^n}{\partial x_{i_{\sigma(n)}} x_{i_{\sigma(n-1)}} \dots x_{i_{\sigma(1)}}}$$

is another such product but *with the factors in a different order.* It is thus seen (although to be completely formal this would require an inductive argument) that, when the n-th order partials exist and are continuous, the factors can be moved around, and the two mixed differential operators give the same result when applied to f. The reason is that we can obtain the permutation σ by a sequence of flips of elements next to each other (the reader familiar with abstract algebra will recognize this as a statement about the symmetric group and two-cycles). Each flip of the partial differential operators does not change the derivative, by the version already proved.

As an example, $f_{xxy} = f_{xyx} = f_{yxx}$ if all the third-order partial derivatives of f exist and are continuous.[3] ∘–•

▶ **[6.47]** : Prove the above result. Also determine the domains of the functions defined in parts (a) through (d).

•–∘ No different from the case of $f : D^1 \to \mathbb{R}$, already proved earlier. One need only repeat the proof, except that the derivatives of f and the arguments of f are in bold. ∘–•

[3]Note that this implies the second-order partials exist and are continuous too, and similarly for the first-order partials.

▶ $[\mathbf{6.48}]$: Prove that the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ are differentiable on $\mathbb{R}^2$.

$a)$ $f(x,y) = x$

$b)$ $f(x,y) = y$

$c)$ $f(x,y) = x^n$ for $n \in \mathbb{Z}^+$

$d)$ $f(x,y) = y^n$ for $n \in \mathbb{Z}^+$

$e)$ $f(x,y) = c_{ij}\, x^i\, y^j$ for $i,j \in \mathbb{Z}^+$ and $c_{ij} \in \mathbb{R}$

$f)$ $f(x,y) = \sum_{1\le i,j\le m} c_{ij}\, x^i\, y^j$

•–○ a) and b) can be verified directly: the derivatives are $[1\ 0]$ and $[0\ 1]$, respectively. c) and d) follow from (b) in the theorem (with induction on n), and d) follows from the linearity of the derivative ((a) and (c) of the theorem). ○–•

▶ $[\mathbf{6.49}]$: Generalize the previous exercise to the case of $f : \mathbb{R}^k \to \mathbb{R}$.

•–○ Any polynomial in $x_1, \ldots, x_k$ (where $\mathbf{x} = (x_1, \ldots, x_k)$ is differentiable. The proof is almost the same (use of the product rule and linearity property). ○–•

▶ $[\mathbf{6.60}]$: Answer the (Why?) question in the above proof.

•–○ Write $\boldsymbol{C} \equiv \boldsymbol{B} - \boldsymbol{A}$. Then $\boldsymbol{C}\mathbf{e_i} = 0$ for all i, so

$$\begin{bmatrix} c_{1,1} & c_{1,2} \ldots c_{1,k} \\ \ldots & \\ c_{p,1} & c_{p,2} \ldots c_{p,k} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} c_{i,1} \\ c_{i,2} \\ \vdots \\ c_{i,k} \end{bmatrix},$$

where the second (column) matrix has the i-th entry nonzero. Thus, all the entries of $\boldsymbol{C}$ are zero. ○–•

▶ $[\mathbf{6.62}]$: Prove the above theorem for the case $f : D^3 \to \mathbb{R}$. Can you prove it for the more general $f : D^k \to \mathbb{R}$?

•–○ Actually, the proof in RC is slightly incomplete. We prove the general case of $f : D^k \to \mathbb{R}$ in detail.

Suppose the partial derivatives $f_i \equiv \frac{\partial f}{\partial x_i}$ all exist in a neighborhood $N_r(\mathbf{a})$ of $\mathbf{a}$ and they are continuous at $\mathbf{a}$. Write $\boldsymbol{A}$ as the Jacobian matrix of partial derivatives. We want to prove that $\boldsymbol{A}$ is in fact the derivative.

Then, if we write $\mathbf{a} = (a_1, \ldots, a_k)$ and fix a point $\mathbf{x} = (x_1, \ldots, x_k)$. We can write $|f(\mathbf{x}) - f(\mathbf{a}) - \boldsymbol{A}(\mathbf{x} - \mathbf{a})|$ as

$$\left| f(x_1, \ldots, x_k) - f(a_1, \ldots, a_k) - \sum_{i=1}^{k} f_i(\mathbf{a})(x_i - a_i) \right|,$$

which we need to bound.

Fix $\epsilon > 0$. We can find a $\delta > 0$ such that $|\mathbf{x}-\mathbf{a}| < \delta$ implies $|f_i(\mathbf{x}) - f_i(\mathbf{a})| < \epsilon$ for each i. (This is by continuity at $\mathbf{a}$, though one must take the min over k different δ_i's for each f_i as in the proof.)

For each i, consider the differences

$$|f(x_1,\ldots,x_i,a_{i+1},\ldots,a_k) - f(x_1,\ldots,x_{i-1},a_i,\ldots a_k) - f_i(\mathbf{a})(x_i-a_i)|\,.$$

By the mean value theorem applied to the function $g(x) \equiv f(x_1,\ldots,x_{i-1},x,a_{i+1},\ldots,a_k) - f_i(\mathbf{a})x$, this is at most

$$|x_i - a_i| \sup_{\mathbf{x},|\mathbf{x}-\mathbf{a}|<\delta} |f_i(\mathbf{x}) - f_i(\mathbf{a})| \leq \epsilon |x_i - a_i|.$$

As a result, we have:

$$|f(x_1,\ldots,x_i,a_{i+1},\ldots,a_k) - f(x_1,\ldots,x_{i-1},a_i,\ldots a_k) - f_i(\mathbf{a})(x_i-a_i)| \leq \epsilon\,|x_i-a_i|\,.$$

If we sum all these equations for each i, we get:

$$|f(x_1,\ldots,x_k) - f(a_1,\ldots,a_k) - \boldsymbol{A}(\mathbf{x}-\mathbf{a})| \leq 2\epsilon \sum_i |x_i - a_i| \leq 2k\epsilon\,|\mathbf{x}-\mathbf{a}|\,.$$

The only problem is that we have $2k\epsilon$ on the right instead of ϵ. But this is not a problem: go back through the proof, and replace ϵ by $\frac{\epsilon}{2k}$ throughout. ○—●

▶ $[\mathbf{6.65}]$: Prove the above theorem.

●—○ By an earlier theorem of this sort, Theorem 2.6, each component f_i is constant (since each component has zero derivative), so $\boldsymbol{f}$ is constant. ○—●

▶ $[\mathbf{6.72}]$: Show that the function $\mathbf{G} : D^p \to \mathbb{R}^m$ in the above proof is continuous at $\mathbf{y} = \mathbf{b}$. Also, answer the (Why?) question following remark number 2.

●—○ $\mathbf{G}$ is clearly continuous everywhere except possibly at $\mathbf{b}$ because it is a quotient of continuous functions and the denominator doesn't vanish (at $\mathbf{y} \neq \mathbf{b}$). But it's continuous at $\mathbf{b}$ too too since the limit of $\mathbf{G}(\mathbf{y})$ as $\mathbf{y} \to \mathbf{b}$ is zero by the definition of the derivative. The (Why?) question is a consequence of the definition and Proposition 3.5. ○—●

▶ $[\mathbf{6.72}]$: Verify (6.35) and therefore that equations (6.36) and (6.37) can be obtained from the one dimensional Taylor's theorem with remainder, Theorem 1.22 on page 252, applied to the function g.

●—○ Write $\mathbf{a} = (a_1,\ldots,a_k)$ and similarly for $\mathbf{x} = (x_1,\ldots,x_k)$. Recall that $g(t) = f(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))$. We saw in the proof of the mean value theorem (Theorem 3.17) that the derivative was

$$g'(t) = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(\mathbf{a} + t(\mathbf{x}-\mathbf{a}))(x_i - a_i).$$

If we compute the n-th derivatives $g^{(n)}(t)$ and prove that they equal $(L_n f)(\mathbf{a} + t(\mathbf{x}-\mathbf{a})$, then we will have proved (6.35), and we will be done by the one-dimensional version of Taylor's theorem with remainder.

We do this by induction on n. Clearly, this is true for $n = 1$, by what has already been shown. We now need to show that

$$\frac{d}{dt}((L_n f)(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) = (L_{n+1} f)(\mathbf{a} + t(\mathbf{x} - \mathbf{a})). \tag{4}$$

But

$$(L_n f)(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) = \sum_{i_1,\dots,i_n} \frac{\partial^n f}{\partial x_{i_n} \dots \partial x_{i_1}}(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) \prod_{j=1}^{n} (x_{i_j} - a_{i_j}).$$

Note incidentally in this expression, $\mathbf{x}$ is to be regarded as a constant, and t as a variable. The derivative of the term

$$\frac{\partial^n f}{\partial x_{i_n} \dots \partial x_{i_1}}(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) \prod_{j=1}^{n} (x_{i_j} - a_{i_j})$$

with respect to t is

$$\sum_{i_{n+1}} \frac{\partial^{n+1} f}{\partial x_{i_{n+1}} \partial x_{i_1} \dots \partial x_{i_n}}(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))(x_{i_{n+1}} - a_{i_{n+1}}) \prod_{j=1}^{n} (x_{i_j} - a_{i_j}),$$

whence a summation establishes (4). ○—●

► **[6.85]** : For each $f(z)$ below, use the difference quotient formulation to verify that the given $f'(z_0)$ is as claimed. Then use the ϵ, δ-version to prove that the derivative is the given $f'(z_0)$.

$a)$	$f(z) = c$ for $D = \mathbb{C}$, constant $c \in \mathbb{C}$,	$f'(z_0) = 0$ for all $z_0 \in D$
$b)$	$f(z) = z^3$ for $D = \mathbb{C}$,	$f'(z_0) = 3z_0^2$ for all $z_0 \in D$
$c)$	$f(z) = z^{-1}$ for $D = \mathbb{C} \setminus \{0\}$,	$f'(z_0) = -z_0^{-2}$ for all $z_0 \in D$
$d)$	$f(z) = z^n$ for $D = \mathbb{C}$, $n \in \mathbb{Z}^+$,	$f'(z_0) = nz_0^{n-1}$ for all $z_0 \in D$
$e)$	$f(z) = z^n$ for $D = \mathbb{C} \setminus \{0\}$, $n \in \mathbb{Z}^-$,	$f'(z_0) = nz_0^{n-1}$ for all $z_0 \in D$
$f)$	$f(z) = \sin z$ for $D = \mathbb{C}$,	$f'(z_0) = \cos z_0$, for all $z_0 \in D$
$g)$	$f(z) = \cos z$ for $D = \mathbb{C}$,	$f'(z_0) = -\sin z_0$, for all $z_0 \in D$

A hint for part f) is to consider the trigonometric identity

$$\sin\theta - \sin\phi = 2\,\cos\left(\frac{\theta + \phi}{2}\right)\,\sin\left(\frac{\theta - \phi}{2}\right).$$

●—○ The proofs are essentially the same as the proofs for the corresponding real functions, which we did earlier (Exercise 6.7). ○—●

► **[6.93]** : Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^3$, and consider the points $z_1 = 1$ and $z_2 = i$ as $\mathbf{p_1}$ and $\mathbf{p_2}$ in the statement of Theorem 3.17 on page 285. Show that there does not exist a complex number playing the role of $\mathbf{q}$ as described in that theorem.

$z_2 - z_1 = i - 1$; $f(z_2) - f(z_1) = -i - 1$; this means we need to look for a point w on $[1, i]$ such that $3w^2 = \frac{-i-1}{i-1} = \frac{(-i-1)(-i-1)}{2} = i$, so $w^2 = \frac{i}{3}$. Checking the two possible values of w shows neither lies on $[1, i]$.

▶ $[\mathbf{6.94}]$: Using the same line of reasoning as in the last example, and recalling the fact that $e^z = e^x \cos y + i\, e^x \sin y$, show that $\frac{d}{dz} e^z$ can be inferred to be e^z.

We have $u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$, so the Jacobian matrix is:

$$\begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

which is of the appropriate form, and which shows the derivative is $e^x \cos x + ie^x \sin y = e^z$.

▶ $[\mathbf{6.96}]$: Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = u(x, y) + i\, v(x, y)$ where $u(x, y) = x^2 + y$ and $v(x, y) = x^2 + y^2$. Infer that the only $z \in \mathbb{C}$ at which $f'(z)$ might exist is $z = -\frac{1}{2} - i\, \frac{1}{2}$.

The Jacobian is easily computed:

$$\begin{bmatrix} 2x & 1 \\ 2x & 2y \end{bmatrix},$$

and if this is to have the appropriate form, $2x = -1$ and $2x = 2y$, whence the claim.

▶ $[\mathbf{6.99}]$: Show that $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{1}{z}$ is differentiable as suggested in the previous example.

Use the quotient rule.

▶ $[\mathbf{6.128}]$: Suppose the norm of $\boldsymbol{A}$ is less than 1. Guess at $\sum_{j=1}^{\infty} \boldsymbol{A}^j$. Can you verify whether your guess is correct?

The answer is $\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{A})^{-1}$ (as we will see, $\boldsymbol{I} - \boldsymbol{A}$ is invertible). The reason is that the sum converges (take norms and use the geometric series), and we have (where the summation starts at zero, not one!)

$$(\boldsymbol{I} - \boldsymbol{A}) \sum_{j=0}^{\infty} \boldsymbol{A}^j = \boldsymbol{I}$$

as one can check by a telescoping argument and the fact that $\boldsymbol{A}^j \to 0$ as $j \to \infty$.

▶ $[\mathbf{6.134}]$: Answer the (Why?) question in the above proof. Also, given a neighborhood $N_\rho(\mathbf{x}_0, \mathbf{y}_0) \subset \mathbb{R}^{k+l}$, show that it contains a product of two neighborhoods $N_{r_1}(\mathbf{x}_0) \times N_{r_2}(\mathbf{y}_0)$.

The (Why?) question follows by looking at the coordinatewise expression $\boldsymbol{G}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \boldsymbol{F}(\mathbf{x}, \mathbf{y}))$; this easily implies that the Jacobian of $\boldsymbol{G}$ can be obtained as in the proof from the "partials" of $\boldsymbol{F}$. For the latter, take $r_1 = r_2 \equiv \frac{\rho}{2}$, for instance. The details are left to the reader.

chapter 7

▶ $[\mathbf{7.2}]$: In what follows, let $f:[a,b]\to\mathbb{R}$ be bounded.

a) Suppose $f:[a,b]\to\mathbb{R}$ is bounded, and that $\mathcal{P}=\{a=x_0,x_1,\dots,x_n=b\}$ is a partition of $[a,b]$. Fix any $0<j<n$, and let $\mathcal{P}_1=\{a=x_0,\dots,x_j\}$ and $\mathcal{P}_2=\{x_j,\dots,x_n=b\}$, so that $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$. Show that $\overline{\mathcal{S}}_{\mathcal{P}_1\cup\mathcal{P}_2}(f)=\overline{\mathcal{S}}_{\mathcal{P}_1}(f)+\overline{\mathcal{S}}_{\mathcal{P}_2}(f)$, and that $\underline{\mathcal{S}}_{\mathcal{P}_1\cup\mathcal{P}_2}(f)=\underline{\mathcal{S}}_{\mathcal{P}_1}(f)+\underline{\mathcal{S}}_{\mathcal{P}_2}(f)$.

b) Consider any bounded function $f:[a,b]\to\mathbb{R}$ and suppose $c\in(a,b)$. If $\mathcal{P}_1=\{a=x_0,x_1,\dots,x_m=c\}$ partitions $[a,c]$ and $\mathcal{P}_2=\{c=x_m,\dots,x_{m+n}=b\}$ partitions $[c,b]$, then $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$ is a partition of $[a,b]$. Show that $\overline{\mathcal{S}}_{\mathcal{P}_1\cup\mathcal{P}_2}(f)=\overline{\mathcal{S}}_{\mathcal{P}_1}(f)+\overline{\mathcal{S}}_{\mathcal{P}_2}(f)$, and that $\underline{\mathcal{S}}_{\mathcal{P}_1\cup\mathcal{P}_2}(f)=\underline{\mathcal{S}}_{\mathcal{P}_1}(f)+\underline{\mathcal{S}}_{\mathcal{P}_2}(f)$.

Note that

$$\overline{\mathcal{S}}_{\mathcal{P}_1\cup\mathcal{P}_2}(f)=\sum_{i=1}^{n}M_i\Delta x_i=\sum_{i=1}^{j}M_i\Delta x_i+\sum_{i=j+1}^{n}M_i\Delta x_i=\overline{\mathcal{S}}_{\mathcal{P}_1}(f)+\overline{\mathcal{S}}_{\mathcal{P}_2}(f).$$

The case for the lower sums is similar. This proves a). Part b) is just a special case of a).

▶ $[\mathbf{7.3}]$: Prove the above lemma.

Say $\mathcal{P}=\{x_0,\dots,x_n\}$. Then, because $m_j\le f(c_j)\le M_j$ for each j, we have:

$$\underline{\mathcal{S}}_{\mathcal{P}}(f)=\sum_{j=1}^{n}m_j\Delta x_j\le\sum_{j=1}^{n}f(c_j)\Delta x_j\le\sum_{j=1}^{n}M_j\Delta x_j=\overline{\mathcal{S}}_{\mathcal{P}}(f),$$

by the basic properties of sups and infs. The middle term is just $\mathcal{S}_{\mathcal{P}}(f,\mathcal{C})$.

▶ $[\mathbf{7.4}]$: Complete the proof of the above lemma by showing that $\underline{\mathcal{S}}_{\mathcal{P}}(f)\le\underline{\mathcal{S}}_{\mathcal{P}'}(f)$.

Note that it is only necessary to consider a 'one point' refinement, since the general case can then be done by induction. To this end, suppose $\mathcal{P}=\{x_0,x_1,\dots,x_n\}$ and $\mathcal{P}'=\mathcal{P}\cup\{\xi\}$, where $x_{k-1}<\xi<x_k$ for some $1\le k\le n$.

Then,

$$\overline{S}_{\mathcal{P}}(f) = \sum_{j=1}^{n} m_j \Delta x_j = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n,$$

and

$$\begin{aligned}\underline{S}_{\mathcal{P}'}(f) = m_1 \Delta x_1 + \cdots + m_{k-1}\Delta x_{k-1} + m_k^{left}(\xi - x_{k-1}) + m_k^{right}(x_k - \xi) \\ + m_{k+1}\Delta x_{k+1} + \cdots + m_n \Delta x_n,\end{aligned}$$

where

$$m_k^{left} \equiv \inf_{[x_{k-1},\,\xi]} f \text{ and } m_k^{right} \equiv \inf_{[\xi,\,x_k]} f.$$

Therefore,

$$\underline{S}_{\mathcal{P}}(f) - \underline{S}_{\mathcal{P}'}(f) = m_k \Delta x_k - \Big(m_k^{left}(\xi - x_{k-1}) + m_k^{right}(x_k - \xi) \Big). \tag{5}$$

To complete the proof, we note that

$$m_k^{left} = \inf_{[x_{k-1},\xi]} f \geq \inf_{I_k} f = m_k, \text{ and similarly, } m_k^{right} \geq m_k.$$

This combined with (5) above yields

$$\begin{aligned}\underline{S}_{\mathcal{P}}(f) - \underline{S}_{\mathcal{P}'}(f) &\leq m_k \Delta x_k - \Big(m_k(\xi - x_{k-1}) + m_k(x_k - \xi) \Big) \\ &= m_k \Delta x_k - m_k \Delta x_k \\ &= 0.\end{aligned}$$

That is,

$$\underline{S}_{\mathcal{P}'}(f) \leq \underline{S}_{\mathcal{P}}(f).$$

(The reader may have noticed that this is word-for-word the same proof as in the text with sups changed to infs, inequalities reversed, and M_j changed to m_j.) ○—●

▶ $[\mathbf{7.6}]$: Suppose $f, g : [a,b] \to \mathbb{R}$ are both bounded functions such that $f(x) \leq g(x)$ on $[a,b]$. Prove that $\underline{\int_a^b} f(x)\,dx \leq \underline{\int_a^b} g(x)\,dx$, and $\overline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} g(x)\,dx$.

●—○ It is clear that $\overline{S}_{\mathcal{P}}(f) \leq \overline{S}_{\mathcal{P}}(g)$ for each partition $\mathcal{P}$ (to see this write out the sums and note that the $M_j^{(f)}$ for f are $\leq$ the corresponding sups $M_j^{(g)}$ for g). Now take the inf over all $\mathcal{P}$. This proves the inequality for the upper integral, and the one for the lower integral is proved similarly. ○—●

▶ $[\mathbf{7.7}]$: Complete the above proof by writing up the similar argument for the lower integral. What δ must ultimately be used to establish the overall result?

●—○ We won't rewrite the whole argument but will suggest another approach: replace f by $-f$. Note that $\underline{S}_{\mathcal{P}}(f) = -\overline{S}_{\mathcal{P}}(-f)$, in fact, by elementary properties of sups and infs (this is essentially the fact that when one multiplies both sides of an inequality by -1 the sign must be reversed). Now similar identities follow for the upper and lower integrals, and thus the part of the theorem for lower integrals is a corollary of the part for upper integrals. We leave the details to the reader. ○—●

▶ $[\mathbf{7.10}]$: For the function of the previous example, show that $\underline{\int_0^1} f(x)\,dx = 0$, and hence, that f is not integrable on $[0,1]$.

•–◦ If any partition $\mathcal{P}$ of $[0,1]$ is chosen, it follows from the density of the rationals that $m_j = 0$ for all j so $\underline{S}_{\mathcal{P}}(f) = 0$. Since $\mathcal{P}$ was arbitrary, this establishes the result. ◦–•

▶ $[\mathbf{7.11}]$: Answer the two (Why?) questions in the above proof.

•–◦ This is because the upper and lower integrals are the inf and the sup of the upper and lower sums (cf. Chapter 2, Proposition 1.14). ◦–•

▶ $[\mathbf{7.12}]$: Verify that $\sqrt{f}$ is bounded on $[a,b]$, and answer the (Why?) question in the above proof.

•–◦ f being bounded by $M \geq 0$ implies $\sqrt{f}$ is bounded by $\sqrt{M}$. The inequality $\sqrt{b} - \sqrt{a} \leq \sqrt{b-a}$ for $0 \leq a \leq b$ is elementary (square both sides). ◦–•

▶ $[\mathbf{7.13}]$: Answer the (Why?) question in the above proof.

•–◦ The max-min theorem lets us find appropriate a_j, b_j such that $f(a_j) = M_j$, $f(b_j) = m_j$. ◦–•

▶ $[\mathbf{7.15}]$: Answer the (Why?) in the above proof, and finish the proof by handling the case where f is nonincreasing.

•–◦ $f(a) = f(b)$ implies f is constant, which answers the (Why?) question. The nonincreasing case is only (essentially) a repetition of the above proof and is left to the reader (alternatively, you could use the fact that f is integrable iff $-f$ is integrable, which is relatively easy to prove from the definitions, and will be proved in Theorem 2.7 below.) ◦–•

▶ $[\mathbf{7.16}]$: Prove the case with $x_0 = a$ or $x_0 = b$, as well as the case of more than one discontinuity in the above theorem. What can happen if there are infinitely many discontinuities in $[a,b]$?

•–◦ If $x_0 = a$, take δ very small and note that f is integrable on $[a+\delta, b]$ as a continuous function there, and any upper/lower sums on $[a, a+\delta]$ are very small (less than $\Lambda\delta$). Thus if $\mathcal{P}$ is a partition of $[a+\delta, b]$ such that

$$\overline{S}_{\mathcal{P}}(f) - \underline{S}_{\mathcal{P}}(f) < \epsilon$$

(where $\epsilon > 0$ is small) then take $\mathcal{P}' = \mathcal{P} \cup \{a\}$.

Now $\mathcal{P}'$ is a partition of $[a,b]$, and it follows easily as in the proof that

$$\overline{S}_{\mathcal{P}'}(f) - \underline{S}_{\mathcal{P}'}(f) = \overline{S}_{\mathcal{P}}(f) - \underline{S}_{\mathcal{P}}(f) + \delta\left(\sup_{[a,a+\delta]} f - \inf_{[a,a+\delta]} f\right) \leq \epsilon + 2\Lambda\delta.$$

If $\delta < \frac{\epsilon}{2\Lambda}$ and $\mathcal{P}$ and $\mathcal{P}'$ are then chosen appropriately, we have:

$$\overline{S}_{\mathcal{P}}(f) - \overline{S}_{\mathcal{P}}(f) < 2\epsilon,$$

whence the result (though you need to replace ϵ with $\epsilon/2$ in the above proof if you want to come out with ϵ at the end). The case $x_0 = b$ is similar, and the case of finitely many discontinuities is an extension of the above with additional small intervals enclosing each of the discontinuities. If there are infinitely many discontinuities f need no longer be integrable, e.g. $f = \chi_{\mathbb{Q}}$. (Cf. Example 1.18). ○—●

▶ $[\mathbf{7.17}]$: Complete the proof of part (a) by showing that kf is bounded on $[a, b]$, and that $\lim\limits_{\|\mathcal{P}\|\to 0} \mathcal{S}_{\mathcal{P}}(k\,f, \mathcal{C}) = k \lim\limits_{\|\mathcal{P}\|\to 0} \mathcal{S}_{\mathcal{P}}(f, \mathcal{C})$.

●—○ f being bounded by M implies kf is bounded by $|k|\,M$. The assertion about the limit follows similarly as in Chapter 4 (or even 3) and the reader should review those proofs as a hint. ○—●

▶ $[\mathbf{7.18}]$: Complete the proof of part (b) by showing that $f \pm g$ is bounded on $[a, b]$ and that $\lim\limits_{\|\mathcal{P}\|\to 0} \mathcal{S}_{\mathcal{P}}(f \pm g, \mathcal{C}) = \lim\limits_{\|\mathcal{P}\|\to 0} \mathcal{S}_{\mathcal{P}}(f, \mathcal{C}) \pm \lim\limits_{\|\mathcal{P}\|\to 0} \mathcal{S}_{\mathcal{P}}(g, \mathcal{C})$.

●—○ f being bounded by M_1 and g being bounded by M_2 imply $f+g$ is bounded by $M_1 + M_2$. The assertion about the limit follows similarly as in Chapter 4 (or even 3) and the reader should review those proofs as a hint. ○—●

▶ $[\mathbf{7.19}]$: Answer the (Why?) question in the above proof, and prove the case for f^-.

●—○ If f is nonnegative on I_j, then $M_j = M_j^+$ and $m_j = m_j^+$ clearly and the inequality is clear. If f is nonpositive on I_j then $M_j^+ = m_j^+ = 0 \le M_j - m_j$ (since $f^+ \equiv 0$)). If f takes both positive and negative values on I_j then $M_j^+ = M_j$, $m_j^+ = \max(m_j, 0) \ge m_j$ and the assertion follows in this case too.

To prove that f^- is integrable, either modify the given proof or use $f^- = f^+ - f$ and the earlier results already proved. ○—●

▶ $[\mathbf{7.20}]$: Verify that fg is bounded on $[a, b]$, and answer the (Why?) question in the above proof. In particular, show that $M_j^{fg} \le M_j^f\, M_j^g$, and that $m_j^{fg} \ge m_j^f\, m_j^g$.

●—○ f's being bounded by M_1 and g's being bounded by M_2 together imply that fg is bounded by $M_1 M_2$.

Note that for $x \in I_j$, $(fg)(x) = f(x)g(x) \le M_j^f M_j^g$. Take the sup over x to get $M_j^{fg} = M_j^f M_j^g$. Note that strict inequality might hold (e.g. $f = 1$ for $0 \le x \le 1$, 0 otherwise; and $g = 1$ for $1 < x \le 2$, 0 otherwise). The case for the infimums is similar. ○—●

▶ $[\mathbf{7.21}]$: Prove the above theorem, noting that $f(x) = f^+(x) - f^-(x)$, and $g(x) = g^+(x) - g^-(x)$.

●—○ Now

$$fg = (f^+ - f^-)(g^+ - g^-) = f^+g^+ + f^-g^- - (f^+g^- + f^-g^+). \tag{6}$$

Each of f^+, f^-, g^+, g^- is integrable. Each product term in the right-hand side of (6) is integrable by Lemma 2.9, and using the linearity of the integral we get the theorem. ○—●

▶ $[\mathbf{7.22}]$: Complete the proof of part (a) by considering the bounded function $h : [a, b] \to \mathbb{R}$ satisfying $h(x) \geq 0$ on $[a, b]$. Show that $\lim_{\|\mathcal{P}\| \to 0} \mathcal{S}_{\mathcal{P}}(h, \mathcal{C}) \geq 0$. Complete the proof of part (b).

•–∘ The completion of the proof of (a) is a simple corollary of the fact that $\mathcal{S}_{\mathcal{P}}(f, \mathcal{C}) \geq 0$ for all $\mathcal{P}, \mathcal{C}$. Cf. the corresponding proofs in Chapters 3-4.

Since $f(x) \leq |f(x)|$ it follows that

$$\int_a^b f(x)dx \leq \int_a^b |f(x)|\, dx,$$

by (a). Similarly, since $-f(x) \leq |f(x)|$, we have

$$-\int_a^b f(x)dx = \int_a^b -f(x)dx \leq \int_a^b |f(x)|\, dx,$$

which now implies (b). ∘–•

▶ $[\mathbf{7.23}]$:
Prove the above corollary.

•–∘ Follows from (b) and (a) of the above theorem. We have:

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\; dx \leq \Lambda\, (b - a).$$

We used the bound on $|f|$ and the fact that

$$\int_a^b \Lambda dx = \Lambda(b - a).$$

∘–•

▶ $[\mathbf{7.24}]$: In this exercise we will establish a result known as **Jordan's Inequality**. It will be useful to us in Chapter 9. In particular, we will establish that $\int_0^\pi e^{-R \sin\theta} d\theta \leq \frac{\pi}{R}$. From this, we will be able to conclude that $\lim_{R\to\infty} \int_0^\pi e^{-R\sin\theta} d\theta = 0$. To begin, note that $\int_0^\pi e^{-R\sin\theta} d\theta = 2 \int_0^{\pi/2} e^{-R\sin\theta} d\theta$. Draw a graph of the sine function on $[0, \pi/2]$ to see that $\sin\theta \geq \frac{2\theta}{\pi}$ on that interval. Verify this inequality analytically by considering the function[1] $f(\theta) = \sin\theta - \frac{2}{\pi}$ on the interval $[0, \pi/2]$. Where does it achieve its maximum and minimum values? With this inequality established, note that

$$\int_0^\pi e^{-R\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{R}(1 - e^{-R}) \leq \frac{\pi}{R}.$$

•–∘ The only part of the solution that remains to be given is the proof that $\sin\theta \geq \frac{2\theta}{\pi}$ on $[0, \pi/2]$. For this, we let $f(\theta) = \sin\theta - \frac{2\theta}{\pi}$. Note that $f(0) = f(\pi/2) = 0$. We shall look for the minimum of f on $[0, \pi/2]$. The critical point is where $\cos\theta = \frac{2}{\pi}$. The second derivative test shows that $f''(\theta) = -\sin\theta < 0$ on $(0, \pi/2)$, so the critical point (whose exact location is irrelevant) is necessarily a maximum. In particular, the minima of f on this interval must occur at the endpoints. So $f(\theta) \geq 0$ on $[0, \pi/2]$, proving the claim. ∘–•

[1] There is a typo in the text here; it should read $\sin\theta - \frac{2\theta}{\pi}$.

▶ $[\mathbf{7.26}]$: Prove the above theorem. Show also that if $c_1 \le c_2 \le \cdots \le c_m$ are points each lying within $[a, b]$, then $\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_m}^b f(x)\,dx$.

•–○ We start with the case $m = 1$ (which is the above theorem). The equality for *upper and lower integrals* was already proved (Lemma 1.15). Since f is known to be integrable on $[a, b]$, and also on each subinterval (Lemma 2.13), we can replace the upper and lower integrals by integrals. The general case of $m \in \mathbb{N}$ follows by induction. ○–•

▶ $[\mathbf{7.28}]$: If F is an antiderivative for $f : [a, b] \to \mathbb{R}$ on $[a, b]$, is it unique? If F and G are two antiderivatives for f on $[a, b]$, how might they differ? Show this by proper use of Definition 2.16.

•–○ Any two antiderivatives of the same function differ by a constant since if F, G are antiderivatives, $(F - G)' \equiv 0$. Conversely if F is an antiderivative of f, so is $F + c$ for any constant c. ○–•

▶ $[\mathbf{7.29}]$: Answer the (Why?) question in the above proof. If f has more than one antiderivative on $[a, b]$, does it matter which one you use in evaluating the integral of f over $[a, b]$?

•–○ The existence of the appropriate points $c_j \in I_j$ follows from the mean value theorem. It doesn't matter which antiderivative you use, because any two differ by a constant (cf. the solution to the previous exercise). ○–•

▶ $[\mathbf{7.33}]$: For a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions of the above definition, show that the sum $\int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ is independent of the choice of c.

•–○ This is a simple consequence of the elementary properties of the integral. First, note that

$$\begin{aligned}\int_a^\infty f(x)dx - \int_b^\infty f(x)dx &= \lim_{d\to\infty}\int_a^d f(x)dx - \lim_{d\to\infty}\int_b^d f(x)dx \\ &= \lim_{d\to\infty}\left(\int_a^d f(x)dx - \int_b^d f(x)dx\right) \\ &= \lim_{d\to\infty}\int_a^b f(x)dx \\ &= \int_a^b f(x)dx.\end{aligned}$$

There is a similar identity for integrals from $-\infty$ to a, b and the proof is similar. The identity runs

$$\int_{-\infty}^a f(x)dx - \int_{-\infty}^b f(x)dx = \int_b^a f(x)dx.$$

We leave the proof to the reader.

Now, fix $c_1, c_2 \in \mathbb{R}$, and consider the difference:

$$\int_{-\infty}^{c_1} f(x)\,dx + \int_{c_1}^{\infty} f(x)\,dx - \int_{-\infty}^{c_2} f(x)\,dx - \int_{c_2}^{\infty} f(x)\,dx.$$

This is the difference of two expressions defining $\int_{-\infty}^{\infty} f(x)dx$ for c_1, c_2. We must prove that it is zero. But, by what we have just seen, it is (with obvious notation)

$$\left(\int_{-\infty}^{c_1} - \int_{-\infty}^{c_2}\right) + \left(\int_{c_1}^{\infty} - \int_{c_2}^{\infty}\right) = \int_{c_2}^{c_1} + \int_{c_1}^{c_2} = 0.$$

○—●

▶ $[\mathbf{7.40}]$: Find $\lim\limits_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n} e^{j/n}$.

●—○ It equals

$$\int_0^1 e^x dx = e - 1.$$

○—●

▶ $[\mathbf{7.42}]$: Complete the above proof by establishing that f is bounded on $[a,b]$.

●—○ Choose N so large that $|f_N(x) - f(x)| < 1$ for all $x \in [a,b]$; then if M is a bound for f_N, $M+1$ is a bound for f by the triangle inequality. ○—●

▶ $[\mathbf{7.44}]$: Let $f_n : [-1,1] \to \mathbb{R}$ be a sequence of functions given by $f_n(x) = \frac{n\,x^2}{n+1+x}$.

a) Find $\lim\limits_{n\to\infty} f_n(x)$.

b) Is the convergence uniform?

c) Compute $\lim\limits_{n\to\infty} \int_{-1}^{1} f_n(x)\,dx$ and $\int_{-1}^{1} \big(\lim\limits_{n\to\infty} f_n(x)\big)\,dx$. Are they equal? Why?

●—○ The limit is x^2. In fact,

$$f_n(x) = x^2 \frac{n}{n+1+x}.$$

The convergence is uniform on $[-1,1]$ because $\frac{n}{n+1+x} \to 1$ uniformly for $x \in [-1,1]$ (This can be proved from the identity

$$\frac{n}{n+1+x} - 1 = \frac{-1-x}{n(n+1)},$$

as it is easy to see that this difference tends to zero uniformly for $x \in [-1,1]$.)

The rest is left to the reader. ○—●

▶ $[\mathbf{7.45}]$: Prove the above corollary.

●—○ Theorem 3.10 applied to the partial sums. ○—●

▶ $[\mathbf{7.46}]$: If the function $\boldsymbol{F} : [a,b] \to \mathbb{R}^p$ in the above definition is given by $\boldsymbol{F}(x) = \big(F_1(x), \ldots, F_p(x)\big)$ where $F_j : [a,b] \to \mathbb{R}$ for $1 \le j \le p$, show that each F_j satisfies $F_j' = f_j$ on $[a,b]$, and is therefore an antiderivative of f_j on $[a,b]$ for $1 \le j \le p$.

•–∘ This follows because the derivative of a vector-valued function is taken componentwise. That is, if $\boldsymbol{F} = (f_1, \ldots, f_p)$, then $\boldsymbol{F}' = (f_1', \ldots, f_p')$. ∘–•

► $[\mathbf{7.47}]$: Prove the above theorem.

•–∘ In this exercise, as in many of the ones below, the result follows their analogues on the real line (see also Exercise 7.46). ∘–•

► $[\mathbf{7.49}]$: Parametrize the circle described in the previous example so that it is traversed in the clockwise direction.

•–∘ $\mathbf{x}(t) = (R\cos(-t), R\sin(-t))$ for $t \in [0, 2\pi]$. ∘–•

► $[\mathbf{7.50}]$: Suppose C in $\mathbb{R}^k$ is the straight line segment which connects two points $\boldsymbol{p} \neq \boldsymbol{q}$ in $\mathbb{R}^k$. Find a parametrization of this "curve."

•–∘ Write $\mathbf{x} : [0,1] \to \mathbb{R}^k$ given by $\mathbf{x} = x\boldsymbol{q} + (1-x)\boldsymbol{p}$. ∘–•

► $[\mathbf{7.55}]$: Confirm the claim in the above example. Then reparametrize the curve so that an imagined particle traverses it three timesmore slowly than the original parametrization.

•–∘ It is easy to see that $\tilde{\tilde{x}}$ is indeed a reparametrization of C because the map $x \to 2x$ is a bijection of $[0, \pi]$ onto $[0, 2\pi]$, and the assertion of the speeds is shown by taking derivatives and using the chain rule. ∘–•

► $[\mathbf{7.57}]$: Prove the above theorem. Begin by considering the case where C is a smooth curve.

•–∘ This is simply a matter of checking the definitions. For instance, the assertions of the terminal and initial points are clear: x_{-C} starts at the terminal point of C and ends at the initial point of C. The piecewise differentiability follows from the Chain Rule. ∘–•

► $[\mathbf{7.59}]$: Let $f, g : D^k \to \mathbb{R}$ be continuous on the open set D^k, and suppose C is a contour in D^k. Show the following:

a) $\int\limits_C (f \pm g)(\mathbf{x})\, dx_j = \int\limits_C f(\mathbf{x})\, dx_j \pm \int\limits_C g(\mathbf{x})\, dx_j$ for each $1 \leq j \leq k$.

b) For any $\alpha \in \mathbb{R}$, $\int\limits_C \alpha\, f(\mathbf{x})\, dx_j = \alpha \int\limits_C f(\mathbf{x})\, dx_j$ for each $1 \leq j \leq k$.

•–∘ Consequences of the linearity property of the regular integral. For instance, if C is a smooth curve with parametrization $\mathbf{x} : [a,b] \to D^k$, then

$$\begin{aligned}\int_C (f+g)dx_j &= \int_a^b (f+g)(\mathbf{x}(t))x_j'(t)dt \\ &= \int_a^b f(\mathbf{x}(t))x_j'(t)dt + \int_a^b g(\mathbf{x}(t))x_j'(t)dt \\ &= \int_C f dx_j + \int_C g dx_j.\end{aligned}$$

∘–•

▶ $[\mathbf{7.61}]$: Prove the above theorem, considering first the case where C is a smooth curve.

•–∘ By the change-of-variables theorem and the chain rule, where g is as in Definition 4.14,

$$\begin{aligned}\int_C f(x_1, \ldots, x_k) dx_j &= \int_a^b f(x_1(t), \ldots, x_k(t)) x_j'(t) dt \\ &= \int_c^d f(x_1(g(u)), \ldots, x_k(g(u))(x_j \circ g)'(t) dt \\ &= \int_C f(\tilde{x_1}, \ldots, \tilde{x_k}) d\tilde{x_j}.\end{aligned}$$

This proceeds under the assumption C is smooth and the general case follows by linearity. ∘–•

▶ $[\mathbf{7.63}]$: Prove the above theorem.

•–∘ Consider the real-valued function $G(t) = F(\mathbf{x}(\mathbf{t}))$. The derivative at t is

$$\boldsymbol{\nabla} F(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt} = \boldsymbol{f}(\mathbf{x}) \cdot \frac{d\mathbf{x}}{dt},$$

Now apply the fundamental theorem of calculus and the above discussion, getting

$$\int_C \boldsymbol{f}(\mathbf{x}) \cdot d\mathbf{x} = G(b) - G(a) = F\left(\mathbf{x}(b)\right) - F\left(\mathbf{x}(a)\right).$$

∘–•

chapter 8

▶ [8.1] : Prove the above proposition.

We prove a), b), and c) by unwinding the definitions. First, it is clear that

$$\int_a^b (z_1 w_1(t) + z_2 w_2(t))\, dt = \int_a^b z_1 w_1(t) dt + \int_a^b z_2 w_2(t) dt,$$

since $Re(z+w) = Re(z) + Re(w)$, and $Im(z+w) = Im(z) + Im(w)$, for $z, w \in \mathbb{C}$. It remains only to prove that

$$\int_a^b z_1 w_1(t) dt = z_1 \int_a^b w_1(t) dt,$$

the same argument applying to the other integral.

This is a long but straightforward computation. Let us write $z_1 = x_1 + iy_1$ and $w_1(t) = c_1(t) + id_1(t)$, where $x_1, y_1, c_1(t), d_1(t)$ are real-valued. We have, then, in view of the linearity property of the real integral:

$$\begin{aligned}
\int_a^b z_1 w_1(t) dt &= \int_a^b (x_1 + iy_1)(c_1(t) + id_1(t)) dt \\
&= \int_a^b (x_1 c_1(t) - y_1 d_1(t)) + i\,(y_1 c_1(t) + x_1 d_1(t))\, dt \\
&= \int_a^b (x_1 c_1(t) - y_1 d_1(t))\, dt + i \int_a^b (y_1 c_1(t) + x_1 d_1(t))\, dt \\
&= x_1 \int_a^b c_1(t) dt - y_1 \int_a^b d_1(t) dt + i \left(y_1 \int_a^b c_1(t) dt + x_1 \int_a^b d_1(t) dt \right) \\
&= (x_1 + iy_1) \left(\int_a^b c_1(t) dt + i \int_a^b d_1(t) dt \right) \\
&= z_1 \int_a^b w_1(t) dt.
\end{aligned}$$

▶ [8.2] : Prove the above proposition.

•–∘ Take real and imaginary parts and apply the real change-of-variables formula. ∘–•

▶ **[8.3]** : Prove the above proposition.

•–∘ Take real and imaginary parts and apply the usual fundamental theorem of calculus. ∘–•

▶ **[8.4]** : Verify that the parametrized circle $C_r(z_0)$ is a smooth curve, and that the default parametrization described in Definition 1.6 associates a counterclockwise direction to the traversal of $C_r(z_0)$.

•–∘ The function is differentiable (clear) and the derivative is ire^{it}, which never vanishes, so the curve is smooth. To verify the counterclockwise direction, try drawing it (noting that $e^{it} = \cos t + i \sin t$). ∘–•

▶ **[8.6]** : Verify that the parametrized segment $[z_1, z_2]$ is a smooth curve with initial point z_1 and terminal point z_2.

•–∘ The curve is differentiable and the derivative w.r.t. t is $z_2 - z_1 \neq 0$. The assertion about the initial and terminal points is clear. ∘–•

▶ **[8.8]** : Verify that the parametrized polygonal contour $P = [z_0, z_1, \ldots, z_N]$ is a contour.

•–∘ It is the catenation of several smooth curves such that the endpoint of one coincides with the starting point of the next. ∘–•

▶ **[8.9]** : Let $E \subset \mathbb{X}$ be open and connected. Show that E is contour-connected. To do this, fix any point $w_0 \in E$, and consider the set

$$A = \{w \in E : \text{There exists a contour } x : [a, b] \to E \text{ such that } x(a) = w_0 \text{ and } x(b) = w\}.$$

(In fact, one can choose a rectangular contour.) Show that A is open. If A is all of E, there is nothing else to prove, so assume there exists $w_1 \in B \equiv E \setminus A$. Show that B is also open, that $\bar{A} \cap B = A \cap \bar{B} = \varnothing$, and hence that $E = A \cup B$ is therefore disconnected, a contradiction. Hence $E = A$ is contour-connected.

•–∘ We carry out the proof in detail. Fix $w_0 \in E$ and consider $A = \{w \in E : w \text{ can be connected to } w_0 \text{ by a rectangular contour in } E\}$. First we show that A is open. If $w_1 \in A$, select r such that $N_r(w_1) \subset E$ (which we can do, E being open). I claim that $N_r(w_1) \subset A$.

We have a rectangular contour C connecting w_0 to w_1. by the definition of A Now if $w_2 \in N_r(w_1)$, then we can connect w_1 to w_2 by a rectangular contour C' in the neighborhood $N_r(w_1) \subset E$. (Draw a picture in the 2-dimensional case; here you only need one horizontal and one vertical line segment.) The catenation (C, C') joins w_0 to w_2. Whence, $w_2 \in A$. Since w_2 was chosen arbitrarily in a small neighborhood of w_1, we see that A is open.

Now let $B = \{w \in E : w \text{ can't be connected to } w_0 \text{ by a rectangular contour in } E\}$. Clearly $B \subset E$ as well, and $A \cup B = E$, $A \cap B = \emptyset$. We will show that B is open as will, which will imply (A being nonempty, as we shall see) that B is empty by connectedness.

Fix $w_3 \in B$. There is a neighborhood $N_\rho(w_3) \subset E$. Now w_3 can't be connected to w_0 by assumption. Suppose $w_4 \in N_\rho(w_3)$ and $w_4 \notin B$, i.e. $w_4 \in A$ so we can join w_0 to w_4 by (say) C_1, a rectangular contour. Then we can join w_4 to w_3 by another rectangular contour C_2 in $N_\rho(w_3)$ since they both lie in that small disk. (C_1, C_2) connects w_0 to w_3, whence $w_3 \in A$, a contradiction.

We've shown that A is open and B is open. We now show A is nonempty by showing $w_0 \in A$. There exists a neighborhood $N_\delta(w_0) \subset E$. Pick $w_5 \in N_\delta(w_0)$ and consider the contour $([w_0, w_5], [w_5, w_0]) \subset N_\delta(w_0) \subset E$.

By the connectedness of E, B is empty so $A = E$. ○—●

▶ **[8.10]** : In this exercise, you will show that if a set $E \subset \mathbb{X}$ is contour-connected, then it is connected. To do this, suppose $E \subset \mathbb{X}$ has the property that for every pair of points $w_1, w_2 \in E$ there exists a contour $x : [a, b] \to E$ such that $x(a) = w_1$ and $x(b) = w_2$. Assume $E = A \cup B$ where A and B are nonempty and $\bar{A} \cap B = A \cap \bar{B} = \varnothing$. Choose $w_1 \in A$ and $w_2 \in B$, and a contour $x : [a, b] \to E$ such that $x(a) = w_1$ and $x(b) = w_2$. Argue that $x\big([a, b]\big)$ is a connected subset of E, and that $x\big([a, b]\big)$ is completely contained in either A or B, a contradiction.

●—○ The continuous image of a connected set such as $[a, b]$ is connected. In particular, $x([a, b])$ is connected. But if the image of x intersected both A and B, we'd have then $x([a, b]) = A \cap x([a, b]) \cup B \cap x([a, b])$, which contradicts connectedness. The rest of the solution is given in the exercise itself. ○—●

▶ **[8.11]** : Suppose $D \subset \mathbb{C}$ is open and connected. Define the open set $D_0 \equiv D \setminus \{z_0\}$ where $z_0 \in D$. Show that for any pair of points $w_1, w_2 \in D_0$, there exists a contour $z : [a, b] \to D_0$ such that $z(a) = w_1$ and $z(b) = w_2$. Therefore D_0 is connected. Convince yourself that this is also true for $\mathbb{R}^k$ for $k \geq 2$, but not for $\mathbb{R}$.

●—○ A rectangular contour (and remember that we can restrict outselves to rectangular contours!) passing through z_0 can be pushed off slightly to avoid z_0 while remaining in D. Try drawing a picture.

For $\mathbb{R}$, the set $\mathbb{R} - \{0\}$ is not even connected, by contrast! There is no path joining -1 to 1 that does not pass through zero (this is the intermediate value theorem). ○—●

▶ **[8.12]** : Suppose $D \subset \mathbb{C}$ is open and connected. Define the open set $D_n \equiv D \setminus \{z_1, z_2, \ldots, z_n\}$ where $z_j \in D$ for $1 \leq j \leq n$. Show that for any pair of points $w_1, w_2 \in D_n$, there exists a contour $z : [a, b] \to D_n$ such that $z(a) = w_1$ and $z(b) = w_2$. Therefore D_n is connected. Convince yourself that this is also true for $\mathbb{R}^k$ for $k \geq 2$, but not for $\mathbb{R}$.

●—○ Induction, using the preceding exercise. ○—●

▶ **[8.15]** : Let C be the circle described in the previous example. Show that for any integer $n \neq 1$, $\oint\limits_C \frac{dz}{(z-z_0)^n} = 0$.

⊸ A simple computation with the usual parametrization for the circle:

$$\oint_C \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{ire^{it}}{r^n e^{int}} dt = \int_0^{2\pi} ir^{1-n} e^{i(1-n)t} dt = 0.$$

⟜

► $[\mathbf{8.16}]$: Let Log : $\mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the principal branch of the logarithm, i.e., Log$(z) = \ln|z| + i\,Arg(z)$, where $0 \le Arg(z) < 2\pi$. Let C_1 be the parametrized unit circle centered at the origin, and let Γ_ϵ be the contour parametrized by z_ϵ : $[\epsilon, 2\pi - \epsilon] \to \mathbb{C}$ where $z_\epsilon(t) = e^{i\,t}$. Note that since $\Gamma_\epsilon \subset \mathbb{C} \setminus [0, \infty)$ for every $\epsilon > 0$, the function Log(z) is continuous on Γ_ϵ. Compute $\oint\limits_{C_1} \text{Log}(z)\,dz \equiv \lim\limits_{\epsilon\to 0} \int\limits_{\Gamma_\epsilon} \text{Log}(z)\,dz$. What happens to the integral defined above if a different branch of the logarithm is used, say, $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $\log(z) = \ln|z| + i\,\widetilde{Arg}(z)$, where $-\pi \le \widetilde{Arg}(z) < \pi$?

⊸ The contour is parametrized by e^{it}, the Log of which is it. (Note that we took $Arg(z)$ between 0 and 2π.) Thus the integral over Γ_ϵ becomes (since the derivative of e^{it} is ie^{it})

$$\int_\epsilon^{2\pi-\epsilon} i^2 t e^{it} dt,$$

which you can evaluate by integration by parts. (Cf. the supplementary exercises in Chapter 7.) ⟜

► $[\mathbf{8.17}]$: Complete the proof of the above proposition by proving *a)*, and proving *b)* for contours and more general subdivisions of C.

⊸ Part a) follows easily from Proposition 1.1. The general case of b) follows by induction, or by a simple direct proof paralleling the one given. ⟜

► $[\mathbf{8.21}]$: Prove the above proposition. Clearly it also holds if Λ is replaced by any $M > \Lambda$.

⊸ With C parametrized by $z : [a, b] \to \mathbb{C}$ and setting $z(t) = x(t) + iy(t)$, we have:

$$\begin{aligned}
\left|\int_C f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right| \\
&\le \int_a^b \left|f(z(t))z'(t)\right| dt \\
&\le \Lambda \int_a^b \left|z'(t)\right| dt \\
&= \Lambda \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt \\
&= \Lambda L_C.
\end{aligned}$$

⟜

► $[\mathbf{8.22}]$: Prove the above proposition. Refer to Proposition 1.3 on page 389.

•–∘ For definiteness suppose C is a smooth curve parametrized by z and $\tilde{z} \equiv z \circ g$ where g is an increasing continuously differentiable function with $g(c) = a$, $g(d) = b$. Now by Proposition 1.3 we have:

$$\begin{aligned}\int_C f(z)dz &= \int_a^b f(z(t))z'(t)dt \\ &= \int_a^b f(z(g(t)))z'(g(t))g'(t)dt \\ &= \int_c^d f(\tilde{z}(s))\tilde{z}'(s)ds.\end{aligned}$$

∘–•

▶ $[\mathbf{8.23}]$: Prove the above claim.

•–∘ Make the standard change-of-variables; see Chapter 7. ∘–•

▶ $[\mathbf{8.24}]$: In the above example, suppose S connects two points of C by going through the exterior of C. Does the conclusion of the example still hold? That is, does the integral of f along S still cancel out the integral of f along $-S$?

•–∘ If the function is continuous there, yes. The proof requires no modifications, as it never used the fact that S was contained in the interior of S. ∘–•

▶ $[\mathbf{8.25}]$: In the above example, suppose the contour C is not simple and S is a contour that connects two points of C. Does the conclusion of the example still hold now?

•–∘ Simplicity was never used. The conclusion holds. ∘–•

▶ $[\mathbf{8.27}]$: Show that $\oint\limits_{\triangle_N} f(z_0)\,dz = \oint\limits_{\triangle_N} f'(z_0)\,(z - z_0)\,dz = 0$ by parametrizing $\triangle_N$ and computing.

•–∘ We prove a more general result: If $\triangle$ is a triangle and $z^* \in \mathbb{C}$ is a constant, then:

$$\oint_\triangle zdz = \oint_\triangle z^* dz = 0.$$

From here the conclusion of the exercise follows easily. (The integral $\oint\limits_{\triangle_N} f'(z_0)\,(z - z_0)\,dz$ splits into two.) First, let $\triangle = [z_0, z_1, z_2, z_0]$. Using the standard parametrizations, we have:

$$\begin{aligned}
\oint_{\triangle} z dz &= \int_{[z_0,z_1]} z dz + \int_{[z_1,z_2]} z dz + \int_{[z_2,z_0]} z dz \\
&= \int_0^1 (z_0 + (z_1 - z_0)t)(z_1 - z_0) dt + \int_0^1 (z_1 + (z_2 - z_1)t)(z_2 - z_1) dt \\
&+ \int_0^1 (z_2 + (z_0 - z_2)t)(z_0 - z_2) dt \\
&= z_0(z_1 - z_0) + z_1(z_2 - z_1) + z_2(z_0 - z_2) + \frac{1}{2}\left((z_1 - z_0)^2 + (z_2 - z_1)^2 + (z_0 - z_2)^2\right) \\
&= 0,
\end{aligned}$$

once you expand all the terms out properly.

The second integral $\oint_{\triangle} z^* dz$ can be computed by noting that $\int_{[z_0,z_1]} z^* dz = z^*(z_1 - z_0)$. Repeat the same for the other line segments, and add all three terms. It is easier than the first.

Aside: Later, in Proposition 2.10, we will see a faster way of proving the result of this exercise. ○—●

▶ **[8.28]** : Fill in the details of the above example.

●—○ A picture is better than words here. ○—●

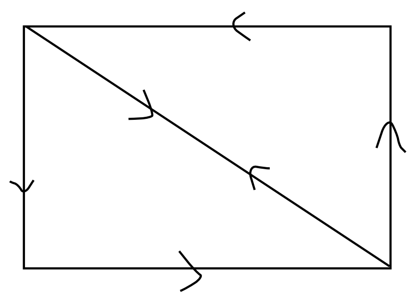

FIGURE 1. The figure in the solution to Ex. 8.28

▶ **[8.29]** : In the above proof, where in cases 2 and 3 is the result of case 1 implicitly used? Extend the corollary to the case where $f : D \to \mathbb{C}$ is differentiable on $D \setminus \{p_1, p_2, \ldots, p_n\}$ for $p_1, p_2, \ldots, p_n \in D$.

●—○ We give a sketch of the generalized case. The extension follows by dividing the first triangle into a lot of small sub-triangles whose interiors contain at most one of the "bad" points $p_1, \ldots, p_n$. Then use the above case to show that the integrals on each small sub-triangle is zero, add the integrals over the sub-triangles to get theintegral over the big triangle, and the integral over the big triangle is zero. ○—●

▶ **[8.32]** : Prove the above theorem for a general closed contour C.

•–∘ Let $C = (C_1, \ldots, C_m)$ be a closed contour parametrized by $z : [a, b] \to \mathbb{C}$. We want to define the function F as in the proof of the theorem:

$$F(s) \equiv \int_a^s \frac{z'(t)}{z(t) - z_0}\, dt.$$

But here z' may not be continuous at certain points (or may not even exist).

Fortunately, there is a way around this: We use the fact that C is *piecewise* smooth. It is straightforward but slightly technical. Note that there exist sub-intervals $[a_j, b_j]$, $1 \leq j \leq m$, overlapping only at endpoints, on the interiors of which z' is continuously differentiable (and nonzero). We order the sub-intervals in the natural way, i.e. so that $b_j = a_{j+1}$. Now fix $s \in [a, b]$ and say the interval $[a_d, b_d] \subset [a, s]$ is the last interval wholly contained in $[a, s]$, or b_d is the largest b_j less than or equal to s. Now F is defined to be

$$F(s) \equiv \left(\sum_{j=1}^{d} \int_{a_j}^{b_j} \frac{z'(t)}{z(t) - z_0}\, dt \right) + \int_{b_j}^{s} \frac{z'(t)}{z(t) - z_0}\, dt.$$

Then it is easy to see that F is continuous (which needs only checking at endpoints b_d, which is easy). Also, by the same calculation, $\frac{e^{F(s)}}{z(s) - z_0}$ has zero derivative on the interior of these intervals $[a_j, b_j]$ and must be constant on them. By continuity, it is constant everywhere, and the same reasoning as in the proof now shows that the winding number is an integer. ∘–•

▶ $[\mathbf{8.34}]$: Answer the (Why?) question in the above proof, and complete the proof by establishing the result for the case where C is a closed parametrized contour.

•–∘ The (Why?) question is answered by noting that the winding number is an integer. The proof really never used the smothness of C (indeed, the bound $\left|\int_C f dz\right| \leq \sup_C |f| L_C$ is still valid for contours) so that no modifications are necessary. ∘–•

▶ $[\mathbf{8.35}]$: In Example 2.5 on page 405, we found that the winding number $n_C(z_0)$ for a counterclockwise circle C around its center z_0 was 1. Consider this same contour again. For $a \in Int(C)$, show that $n_C(a) = 1$. What if $a \in Ext(C)$?

•–∘ The winding number is constant on the connected set $Int(C)$ (cf. Proposition 2.7) and must be 1 on $Int(C)$ since it is 1 on z_0. It is zero on $Ext(C)$ by Proposition 2.8. ∘–•

▶ $[\mathbf{8.37}]$: Prove the above proposition.

•–∘ Assume C is smooth; the general case follows easily by piecing one smooth curve after another. Parametrize C by $z : [a, b] \to \mathbb{C}$. We have:

$$\int_C f(z)\, dz = \int_a^b f(z(t)) z'(t) dt = \int_a^b (f \circ z)' dt$$
$$= F(z(b)) - F(z(a)) = F(z_T) - F(z_I).$$

This proof should not be new. ∘–•

▶ [**8.38**] : Prove the above corollary.

●—○ In a), we have $z_T = z_I$, and the result follows from the fundamental theorem of calculus. The second part is just as straightforward. ○—●

▶ [**8.40**] : Let $D \subset \mathbb{C}$ be star-shaped and suppose $f : D \to \mathbb{C}$ is such that $f' \equiv 0$ on D. Show that $f \equiv c$ on D for some $c \in \mathbb{C}$.

●—○ This is actually true even if D is only connected. It is slightly easier in this case though. Let z_0 be a star-center, pick $z \in D$, define $G(t) = f((1-t)z_0 + tz)$, use the chain rule to show the derivative G' vanishes, show that G is constant as a result, and deduce f is constant on rays emanating from z_0, hence on D. ○—●

▶ [**8.41**] : Prove the above corollary.

●—○ This follows via the same proof but using the modification of the triangle lemma where the function is allowed not to be differentiable at one point (but still required to be continuous). ○—●

▶ [**8.42**] : Prove that the conclusion to Corollary 2.15 still holds even for a finite collection $p_1, \ldots, p_n$ of excluded points.

●—○ One need only show that the triangle lemma still holds. For this, cf. Exercise 8.29. ○—●

▶ [**8.44**] : Prove *(i)* ⇔ *(ii)* in the above theorem.

●—○ If integrals are path-independent, then the integral on a closed contour starting and ending at z_0 is the same as the integral around the constant path at z_0, which is zero. Conversely suppose that integrals around closed contours are 0. If C_1, C_2 are two closed contours starting and ending at the same points then $C \equiv (C_1, -C_2)$ is closed, so f integrates to zero along C, whence the integrals on C_1 and C_2 are equal. ○—●

▶ [**8.47**] : Complete the proof of the above corollary by showing that, if $w_0 \in Ext(C_1)$, then $w_0 \in Ext(C_2)$, implying $n_{C_2}(w_0) = n_{C_1}(w_0) = 0$.

●—○ Since $C_2 \subset Int(C_1)$, it follows that $C_2 \cap Ext(C_1) = \emptyset$, so $Ext(C_1) \subset Ext(C_2)$. (Note that $Ext(C_1)$ is a connected unbounded subset of $\mathbb{C}$ contained in $\mathbb{C} - C_2$, and appeal to the Jordan curve theorem.) The assertion follows. ○—●

▶ [**8.49**] : Show that the function g described in the proof of the above theorem is continuous on D and differentiable on $D \setminus \{z_0\}$.

●—○ Continuity is clear everywhere except possibly at z_0, but we see that it is continuous there too since the limit of g at z_0 is $f'(z_0)$ by definition. Differentiability outside z_0 follows from the quotient rule. ○—●

▶ [**8.50**] : How do we know a disk such as D' as described in each of the cases of the above proof exists? What type of region is D' in each case? To answer these questions, let C_r be a circle such that it and its interior are contained in the open set $D \subset \mathbb{C}$. If r and z_0 are the circle's radius and center, respectively, show that there exists a neighborhood $N_\rho(z_0)$ such that $C_r \subset N_\rho(z_0) \subset D$. (Hint: Assume otherwise, that is, that for all $\rho > r$ we have $N_\rho(z_0) \cap D^C \neq \varnothing$. In particular, show that there exists $w_n \in N_{r+\frac{1}{n}}(z_0) \cap D^C$.)

•–○ Suppose C_r and its interior are contained in D and $D \subset \mathbb{C}$ is open. Then by definition $\overline{N_r(z_0)} \subset D$. I claim that there is $\rho > r$ with $N_\rho(z_0) \subset D$ as well (the reader may wish to draw a picture).

If there is no $\rho > r$ such that $N_\rho(z_0) \cap D^C = \emptyset$, then choose $\rho = r + \frac{1}{n}$ for each n. There exists thus a sequence of points $x_n \in D^C \cap N_{r+\frac{1}{n}}(z_0)$. Each point is not contained in the interior of C_r because that interior is contained in D. Thus it is seen that the sequence of points x_n is contained between C_r and $C_{r+\frac{1}{n}}$. The sequence thus forms a bounded set which has a limit point by the Bolzano-Weierstrass Theorem. This limit point must be on C_r, and it lies in D^C since the latter is a closed set. Thus $C_r \cap D^C \neq \emptyset$, a contradiction. ○–•

► $[8.4]$: Use Corollary 4.2 to prove *Gauss's mean value theorem.* If $f : D \to \mathbb{C}$ is differentiable on the open set $D \subset \mathbb{C}$, and $C_r(z_0)$ and its interior are contained within D, then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\, e^{i\,t})\, dt$.

•–○ We have, by Corollary 4.2 and the usual parametrization of a circle::

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ie^{it} f(z_0 + e^{it})}{e^{it}} dt,$$

whence the result. ○–•

► $[8.58]$: Prove the general case.

•–○ As the general case similar but more tedious, we present the basic idea in a more abstract form. Let G be a differentiable complex function on $\mathbb{C} - \{0\}$ whose derivative is continuous there. Suppose that for every compact K not containing zero and $\epsilon > 0$, there is a $\delta > 0$ such that $z_0 \in K$ and $|z - z_0| < \delta$ implies

$$\left| G(z) - G(z_0) - G'(z_0)(z - z_0) \right| \leq \epsilon \, |z - z_0| \, .$$

This means that G is *uniformly* differentiable on compact sets. Let $f : C \to \mathbb{C}$ be continuous. Define for $z \notin C$

$$F_G(z) = \int_C f(\zeta) G(\zeta - z) d\zeta.$$

I claim then that F_G is differentiable on $\mathbb{C} - C$ and

$$F_G'(z) = -\int_C f(\zeta) G'(\zeta - z) d\zeta.$$

The proposition will then follow in the general case by induction by taking for G a power of $\frac{1}{z}$. It is left to the reader to check the uniform differentiability assumption holds for such G.

We now sketch the proof of the claim. Write:

$$F_G(z) - F_G(z_0) + (z - z_0)\int_C f(\zeta)G'(\zeta - z_0)dz$$
$$= \int_C f(\zeta)(G(\zeta - z) - G(\zeta - z_0) - ((z_0 - z)G'(\zeta - z_0))d\zeta.$$

We can bound this; supposing $|f(\zeta)| \leq M$ on C, we get that

$$\left| F_G(z) - F_G(z_0) + (z - z_0)\int_C f(\zeta)G'(\zeta - z_0)d\zeta \right|$$

is bounded by

$$ML_C \sup_{\zeta \in C} \left| G(\zeta - z) - G(\zeta - z_0) - ((z_0 - z)G'(\zeta - z_0)) \right|.$$

Taking z very close to z_0 the term in the absolute value is uniformly less than a small multiple of $|z - z_0|$ (which equals $|\zeta - z - (\zeta - z_0)|$ for each $\zeta \in C$), by uniform differentiability. Thus we see that

$$\left| F_G(z) - F_G(z_0) + \int_C f(\zeta)G'(\zeta - z_0)d\zeta \right|$$

is bounded by a small multiple of $|z - z_0|$ for z close to z_0, which proves the assertion on the derivative. ○—●

► $[8.59]$: Prove the above corollary to Proposition 4.11.

●—○ f is differentiable in the interior of each simple closed curve by Proposition 4.11. It is thus differentiable at any $z_0 \in D$, since z_0 is in the interior of a small circle centered at z_0. ○—●

► $[8.60]$: Prove the above corollary using Proposition 4.11.

●—○ When $n = 0$ this is just Cauchy's formula. The general case follows by induction, differentiating under the integral sign from Proposition 4.9 (the $n!$ comes from the fact that the n-th differentiation adds the factor n). Finally the bounds follow from the formula and the ML inequality (the standard estimate for integration on contours). In fact, the integrand is bounded by $\frac{\Lambda}{r^{n+1}}$, as is easily seen. ○—●

► $[8.63]$: If $u : D^2 \to \mathbb{R}$ is harmonic on D^2, then u is $C^\infty(D^2)$. In fact, $u = Re(f)$ for some differentiable complex function $f : D \to \mathbb{C}$, where D is D^2 as represented in $\mathbb{C}$. Note that this result only holds locally, even if D is connected. To see this, consider $u : D^2 \to \mathbb{R}$ given by $u(x, y) = \ln(x^2 + y^2)$ where $D^2 = \mathbb{R}^2 \setminus \{0\}$. Determine the harmonic conjugates of u. How do they differ from each other? What is the domain of the resulting differentiable complex function f?

●—○ Note that $\ln(x^2 + y^2) = Re(Log(z^2)) = 2Re(Log(z))$. The harmonic conjugate can thus be taken to be $2Arg(z)$. Note that it is not continuous (let alone differentiable) in the domain of u. The harmonic conjugate can be defined locally as a branch of $2arg(z)$, but not globally in a continuous manner. ○—●

► $[\mathbf{8.64}]$: For fixed z, show that $\lim\limits_{n\to\infty} \oint\limits_{C_r} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta = \oint\limits_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$, and thereby complete the proof of the above theorem.

•—○ This proof uses essentially the same idea as the proof that integration of a limit of a uniformly convergent sequence of functions is the limit of the integrals (Chapter 7). Fix $\epsilon > 0$ and take N so large that $n > N$ implies $|f_n(z) - f(z)| < \epsilon$ if $z \in C_r$. Note that

$$\begin{aligned}
\left| \oint\limits_{C_r} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta - \oint\limits_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| &= \left| \oint\limits_{C_r} \frac{f_n(\zeta) - f(z)}{\zeta - z}\, d\zeta \right| \\
&\le \oint\limits_{C_r} \frac{|f_n(\zeta) - f(z)|}{|\zeta - z|}\, d\zeta \\
&\le \oint\limits_{C_r} \frac{\epsilon}{\min_{\zeta \in C_r} |\zeta - z|}\, d\zeta \\
&= \frac{2\pi\epsilon}{\min_{\zeta \in C_r} |\zeta - z|}.
\end{aligned}$$

This gives the appropriate result, since $\epsilon > 0$ was arbitrary. (If you want, replace ϵ by $\frac{\min_{\zeta \in C_r} |\zeta - z| \epsilon}{4\pi}$ to come out with $< \epsilon$ at the end.) ○—•

► $[\mathbf{8.65}]$: Let $f : D \to \mathbb{C}$ be continuous on the open set $D \subset \mathbb{C}$, and suppose f is differentiable on $D \setminus \{p\}$ for some $p \in D$. Show that f is differentiable on all of D. What if f is presumed initially to only be differentiable on $D \setminus \{z_1, z_2, \dots, z_n\}$ for $z_1, z_2, \dots, z_n \in D$?

•—○ We first prove a stronger version of Morera's theorem, where the integral is only required to vanish on triangles. The proof of Morera's theorem required only that $\oint_\triangle f(z)dz = 0$ for each *triangle* $\triangle$ in D. If you do not believe this, note that as in the theorem we need only show that f is differentiable in a neighborhood of each $z_0 \in D$, if z_0 is arbitrary—in particular, we can reduce to the star-shaped case. We can still prove that $F'(z) = f(z)$ as in the proof of Theorem 2.14 (cf. also Corollary 2.15). In fact, the same reasoning works, except in that theorem the triangle lemma was used to show that $\oint_\triangle f(z)dz = 0$ for each $\triangle$. Here we have assumed it. In particular:

Theorem. *If f is continuous on an open subset of $\mathbb{C}$ and its integral vanishes on all triangles, then f is differentiable.*

So here, to complete the solution, we need only to show that $\oint_\triangle f(z)dz = 0$ for each triangle $\triangle$ in D. We may assume D is a disk (since we need only prove that f is differentiable in each disk). Now the assertion follows from the corollary to the triangle lemma. The reader should note that we assumed D was a disk to ensure that the interior of a triangle in D lay in D. ○—•

chapter 9

▶ $[\mathbf{9.1}]$: Show that when $R = 0$ the associated power series centered at ξ_0 converges absolutely at ξ_0, but diverges for $\xi \neq \xi_0$.

Any power series centered at ξ_0 converges absolutely at ξ_0, since all but possibly one of the terms are zero! If a power series with $R = 0$ converged outside ξ_0, then we would have $R > 0$ by the definition, contradiction.

▶ $[\mathbf{9.2}]$: Suppose $\sum_{j=0}^{\infty} a_j\,(\xi - \xi_0)^j$ is a power series centered at ξ_0 that converges for $\xi_1 \neq \xi_0$. Show that the series converges absolutely for any ξ such that $|\xi - \xi_0| < |\xi_1 - \xi_0|$, that is, for any $\xi \in N_r(\xi_0)$ where $r \equiv |\xi_1 - \xi_0|$.

Indeed, by the definition of R as a supremum it follows that $R \geq |\xi_1 - \xi_0|$, so we can apply the theorem about convergence inside the neighborhood of convergence.

▶ $[\mathbf{9.3}]$: Show that a power series converges uniformly on any compact subset of its neighborhood of convergence.

If $K \subset N_r(x_0)$ is compact, then I claim that in fact $K \subset \overline{N_s(x_0)}$ for some $s < r$; by the theorem, this implies the exercise. To prove the claim, note that $f : K \to \mathbb{R}$ defined by $f(x) \equiv |x - x_0|$ is continuous on K and consequently assumes its upper bound s on K (which s is consequently less than r), and the claim $K \subset \overline{N_s(x_0)}$ is now clear with our choice of s.

▶ $[\mathbf{9.4}]$: Consider the complex power series given by $\sum_{j=0}^{\infty} \frac{1}{j^2}\, z^j$. Use the ratio test to show that $R = 1$, and that the series converges absolutely at every point on its circle of convergence $C_1(0)$.

A little computation (specifically, the fact that $\lim_{j\to\infty} \left(\frac{j+1}{j}\right)^2 = 1$) shows the limit of the ratios of successive terms $\frac{z^j}{j^2}$ and $\frac{z^{j+1}}{(j+1)^2}$ is z, so the ratio test now applies. The limit of successive ratios is < 1 in absolute value precisely when $|z| < 1$, and the limit of successive ratios is > 1 in absolute value precisely when $|z| > 1$. The series thus converges in the unit disk and diverges outside the closed unit disk. The fact that the series converges absolutely on the circle of convergence follows by the convergence of $\sum_j \frac{1}{j^2}$.

▶ $[\mathbf{9.5}]$: Can you find an example of a power series that does not converge at any point on its circle of convergence?

•–◦ Take

$$\sum_j z^j;$$

the ratio test shows the radius of convergence is one, as in the previous exercise, yet it does not converge for any z with $|z| = 1$, as the terms z^j do not tend to zero. ◦–•

▶ $[\mathbf{9.6}]$: Suppose $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ is given by $f(z) = \frac{1}{z-1}$. Find a power series representation for f centered at $z = 2$, and determine its radius and neighborhood of convergence.

•–◦ Write

$$f(z) = \frac{1}{1 + (z-2)} = \sum_{j=0}^{\infty} (-1)^j (z-2)^j$$

by the geometric series; we omit the proof that the radius of convergence is one, which follows from the ratio test (or the geometric series). ◦–•

▶ $[\mathbf{9.7}]$: Choose any real $\alpha > 0$. Can you find a power series having radius of convergence equal to α?

•–◦

$$\sum_j \alpha^{-j} z^j.$$

◦–•

▶ $[\mathbf{9.9}]$: Answer the (Why?) question in the above proof. Then prove the general case; to do so, let $y = x - x_0$ so that $f(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j$ becomes $f(x_0 + y) = \sum_{j=0}^{\infty} a_j y^j$. Then rescale by letting $y = Rt$. This gives $f(x_0 + Rt) = \sum_{j=0}^{\infty} a_j R^j t^j$.

•–◦ Since $x > 0$, $|1 - x| = 1 - |x| = 1 - x$. This answers the (Why?) question. The details of the rescaling are left to the reader, as the procedure is already given in exercise itself. The idea is simply to move x_0 to zero by making a translation, and then make a dilation (scaling) to make the radius one. ◦–•

▶ $[\mathbf{9.10}]$: What conditions need be presumed in the above theorem in order to conclude that $\lim\limits_{x \to (x_0 - R)^+} f(x) = \sum_{j=0}^{\infty} a_j (-R)^j$?

•–◦ By making the transformation $x \to -x$, this exercise just becomes a new form of the theorem; we need

$$\sum a_j (-R)^j$$

to converge. (This is a *sufficient*, but not necessary, condition.) ◦–•

▶ $[\mathbf{9.11}]$: Prove part 1 of the above theorem.

This follows because we can add series, and thus power series, term-by-term. In particular, in the domain where both series converge:

$$\begin{aligned}\sum a_n(z-z_0)^n + \sum b_n(z-z_0)^n &= \sum \left(a_n(z-z_0)^n + b_n(z-z_0)^n\right) \\ &= \sum (a_n+b_n)(z-z_0)^n.\end{aligned}$$

▶ $[9.12]$: Use induction to prove inequality (9.5) for $j \geq 1$.

Suppose the inequality true for indices smaller than j (i.e. we use *complete* induction). Then we have, in view of the inductive assumption

$$|b_j| \leq \frac{1}{|a_0|} \sum_{k=1}^{j} |a_k| \, |b_{j-k}| \leq \frac{1}{|a_0|} \sum_{k=1}^{j} |a_k| \, \frac{2^{j-k}}{|a_0|} \left(\frac{M}{r}\right)^{j-k},$$

so using

$$|a_k| \leq \frac{M\,|a_0|}{r^k}$$

yields (since $M \geq 1$)

$$|b_j| \leq \frac{1}{|a_0|} \sum_{k=1}^{j} \frac{2^{j-k}}{|a_0|} \frac{M\,|a_0|}{r^k} \left(\frac{M}{r}\right)^{j-k} \leq \frac{1}{|a_0|} \left(\frac{M}{r}\right)^{j} \sum_{k=1}^{j} 2^{j-k} \leq \frac{1}{|a_0|} \left(\frac{2M}{r}\right)^{j}.$$

▶ $[9.18]$: Complete the proof of the above theorem by extending the result to higher-order derivatives. Also, show that the original power series and its claimed derivative have the same radius of convergence by applying Theorem 1.9 on page 461.

The proof for higher-order derivatives is simply induction now, since the case of first-order derivatives has already been established. The fact that the radius of convergence of $\sum_j ja_j(z-z_0)^{j-1}$ is the same as that of $\sum_j a_j z^j$ follows from $\lim_j |j|^{1/j} = 1$.

▶ $[9.20]$: Prove the uniqueness in the above theorem.

Evaluate the n-th derivatives at x_0 for each n to evaluate the coefficient of $(x-x_0)^n$.

▶ $[9.27]$: Show directly that the radius of convergence of the series in (9.20) is 1 except in one important case. What is that case?

If c is not a nonnegative integer, then the successive ratios of the coefficients can be computed to have limit 1. Indeed:

$$\frac{\binom{c}{n+1}}{\binom{c}{n}} = \frac{c-n}{n+1} \to 1 \text{ as } n \to \infty.$$

Therefore, by the ratio test, the radius of convergence is one. If c is a nonnegative integer, the coefficients become zero for n sufficiently large, and the radius of convergence is infinite.

▶ $[\mathbf{9.31}]$: Prove the above theorem. (Hint: Consider arbitrary $\xi_1 \in N_r(\xi_0)$. Choose $\rho = \frac{1}{2}(r - |\xi_1 - \xi_0|)$, and consider $N_\rho(\xi_1)$. Show that f has Taylor series representation on $N_\rho(\xi_1)$.)

•–∘ We know that f has a Taylor series representation $f(\xi) = \sum c_j(\xi - \xi_0)^j$ centered at ξ_0 that converges in the disk of radius r. Now consider

$$f(\xi_1 + t) = f(\xi_0 + (\xi_1 - \xi_0 + t)) = \sum c_j(\xi_1 - \xi_0 + t)^j.$$

If we prove that this can be rearranged legitimately into a power series in t converging in a neighborhood of zero, then we will be done. Indeed,

$$\sum c_j(\xi_1 - \xi_0 + t)^j = \sum_j \sum_{k \leq j} c_j t^k \binom{j}{k} (\xi_1 - \xi_0)^{j-k}.$$

For $|t| < \rho$, it is easy to see that this converges absolutely:

$$\sum_j \sum_{k \leq j} |c_j| \, |t|^k \binom{j}{k} |\xi_1 - \xi_0|^{j-k} \leq \sum_j |c_j| \, (|\xi_1 - \xi_0| + \rho)^j$$

by the binomial theorem, and this last converges absolutely because of the power series $\sum c_j w^j$ has radius of convergence r and $|\xi_1 - \xi_0| + \rho < r$. By absolute convergence, we can rearrange the series into a power series $\sum d_j t^j$ (collecting coefficients for each t^j) that therefore converges for $|t| < \rho$. ∘–•

▶ $[\mathbf{9.34}]$: Prove the above proposition.

•–∘ A complex analytic function can always be expanded locally in a Taylor series, by Taylor's theorem for complex functions (Theorem 2.3). Hence it is analytic on its domain. ∘–•

▶ $[\mathbf{9.38}]$: Show that g is differentiable on D.

•–∘ Indeed, this follows because g can be expressed in a power series. ∘–•

▶ $[\mathbf{9.40}]$: Prove the above corollary.

•–∘ Part a) is immediate from the preceding theorem. Part b) follows from it by applying it to $h \equiv f - g$. ∘–•

▶ $[\mathbf{9.43}]$: Prove the above proposition.

•–∘ This follows because the processes of taking the limit and integration can be interchanged in the case of uniform convergence (valid by the previous proposition). Cf. Theorem 3.10 of Chapter 7; that result applies to real integrals, but the generalization for complex line integrals is a corollary. ∘–•

▶ $[\mathbf{9.44}]$: Establish the uniqueness of the b_k terms in the Laurent series representation (9.22) by carrying out the integral $\oint\limits_C \frac{f(\zeta)}{(\zeta - z_0)^{-k+1}} \, d\zeta$ for any fixed $k \geq 1$.

•–∘

$$\oint\limits_C \frac{f(\zeta)}{(\zeta - z_0)^{-k+1}} \, d\zeta = \sum_{j=0}^{\infty} a_j \int_C (\zeta - \zeta_0)^{j+k-1} d\zeta + \sum_{j=1}^{\infty} \int_C b_j (\zeta - \zeta_0)^{-j+k-1} d\zeta$$

If $k = 1$, this integral is $2\pi i a_0$. If $k > 1$, the integral is $2\pi i b_k$ (by an easy integral computation over a circle that you have already probably done before in another form). Hence the $\{b_k\}$ are uniquely determined by f. ○—●

▶ $[\mathbf{9.51}]$: Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = e^{1/z}$. Show that the associated singularity of $f \circ g$ at $z_0 = 0$ is removable, and hence that f has a removable singularity at infinity.

●—○ The Laurent expansion of $f(1/z)$ is that of e^z near zero, namely

$$\sum z^j/j!$$

which shows that $f(1/z)$ has a removable singularity at $z = 0$, hence f has a removable singularity at ∞. ○—●

▶ $[\mathbf{9.53}]$: Complete the proof of part a) of the above theorem.

●—○ If f can be extended to a holomorphic function on a neighborhood of z_0, this extension has a Taylor expansion centered at z_0 (with, obviously, no negative powers of $(z - z_0)$); this must be the Laurent expansion of f around z_0 by uniqueness, and so f has a removable singularity at z_0. ○—●

▶ $[\mathbf{9.57}]$: Prove Theorem 6.3.

●—○ Without loss of generality, assume $z_0 = 0$. Suppose $f_0(z) = \sum_{j \geq 0} c_j z^j$ is the Taylor expansion ; then f has Laurent expansion $\sum_{j \geq -N} c_{j+N} z^j$. We need to compute c_{N-1}; this is the residue of f as it is the coefficient of z^{-1}. But in fact

$$(N-1)! c_{N-1} = f_0^{(N-1)}(z_0)$$

by term-by-term differentiation. This proves the theorem. ○—●

▶ $[\mathbf{9.60}]$: Prove the following: Suppose the functions f and g are differentiable at z_0, $f(z_0) \neq 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$. Then the function given by $\frac{f(z)}{g(z)}$ has a simple pole at z_0, and $\mathrm{Res}\left(\frac{f(z)}{g(z)}, z_0\right) = \frac{f(z_0)}{g'(z_0)}$.

●—○ We can write $g(z) = (z - z_0)h(z)$, where $h(z)$ is analytic and $h(z_0) = g'(z_0)$ by dividing out a factor from the power series. Now

$$q(z) = \frac{f(z)}{h(z)}$$

is analytic in a neighborhood of z_0 with a Taylor series expansion whose lowest term is $\frac{f(z_0)}{g'(z_0)}$. Thus the lowest term in the Laurent expansion of f/g is $\frac{1}{z-z_0}\frac{f(z_0)}{g'(z_0)}$. This proves the assertion about the residue and the simple pole. ○—●

chapter 10

▶ **[10.2]** : Verify the claims made above abou tthe distance function d.

Indeed, this follows from $\hat{F}$'s being a one-to-one correspondence. For instance, if $d(z_1, z_2) = 0$, then

$$\left|\hat{F}^{-1}(z_1) - \hat{F}^{-1}(z_2)\right| = 0$$

so that $\hat{F}^{-1}(z_1) = \hat{F}^{-1}(z_2)$, implying $z_1 = z_2$. This establishes one property of d. The others (e.g. the triangle inequality) are proved similarly.

▶ **[10.7 and 9]** :

A line in $\mathbb{C}$ is given by an equation $ax + by = c$, and this can be put in the form $cz + d\bar{z} = e$. Under the transformation $z \to 1/z$,

$$c\frac{1}{z} + d\frac{1}{\bar{z}} = e$$

which becomes

$$c\bar{z} + dz = e|z|^2.$$

It is easy to see that this is an equation of a circle (if $e \neq 0$) or a line (if $e = 0$).

▶ **[10.10]** : Prove the above. Begin by computing the derivative of $T(z)$.

The derivative may be computed via the quotient rule; it is easy to see that T is differentiable wherever the denominator does not vanish. In addition, it is easy to check that T is continuous on $\hat{\mathbb{C}}$ because linear maps $z \to az + b$ and inversion $z \to \frac{1}{z}$ all are (exercise). To prove that T is a one-to-one correspondence, we can consider an equation $\frac{az+b}{cz+d} = w$ and use elementary algebra(!) to solve for z in terms of w uniquely. Indeed, doing so will prove that the inverse map is an LFT itself.

▶ **[10.11]** : Suppose T is an LFT. If $A \subset \widehat{\mathbb{C}}$ is a line, show that $T(A)$ is a line or a circle. When will it be a circle? If $B \subset \mathbb{C} \subset \widehat{\mathbb{C}}$ is a circle, show that $T(B)$ is a line or a circle. When will it be a line?

•─○ It is enough, by Proposition 2.2, to establish this for the special cases of translation, rotation/dilation, and inversion. These have already been handled (cf. Exercises 10. 5, 10.6, 10.7, 10.9). A line and not a circle is unbounded, so the image will be a line when the original object contains $-d/c$.

We can give a direct proof as follows. A set defined by an equation (with not all constants zero)

$$a + bz + c\bar{z} + dz\bar{z} = 0$$

is either a line (if $d = 0$) or a circle (if $d \neq 0$). It is easy to check directly that an LFT transforms such an equation into another of the same form by clearing denominators. ○─•

▶ $[\mathbf{10.12}]$: Prove the above.

•─○ Direct computation:

$$\frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} = \frac{a(a'z+b') + b(c'z+d')}{a'z+b' + d(c'z+d')}$$

and it is clear that the latter can be written as an LFT after simplification. ○─•

▶ $[\mathbf{10.15}]$: Show that the equation $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ implicitly defines an LFT given by $w = T(z)$ that maps z_1, z_2, and z_3 to w_1, w_2, and w_3, respectively. The expression on each side of the equality implicitly defining w in terms of z is called a **cross ratio**.

•─○ Define the LFTs $A : z \to \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ and $B : z \to \frac{(z-w_1)(w_2-w_3)}{(z-w_3)(w_2-w_1)}$. It is clear that the implicit map $z \to w$ is given by $B^{-1} \circ A$, which is an LFT by Proposition 2.3. The fact that the three points are mapped to each other is easy to check from the definitions. ○─•

▶ $[\mathbf{10.18}]$: Show that the mapping constructed above is onto the open unit disk. Also, show that the imaginary axis is mapped onto the unit circle.

•─○ Suppose w is such that $|w| < 1$. We will find a z in the right half-plane with $\frac{1-z}{1+z} = w$. Indeed, this says that

$$1 - z = w(1+z), \text{ or } (w+1)z = 1 - w, \text{ or } z = \frac{1-w}{1+w}.$$

That is, the map f is its own inverse! We need to check then that if $w \in D_1(0)$, then $f(w)$ is in the right half-plane. But, if $w = a + ib$,

$$Re(z) = Re\left(\frac{1-w}{1+w}\right) = Re\left(\frac{(1-a-ib)(1+a-ib)}{(1+a)^2+b^2}\right).$$

We need only check that the numerator has positive real part. This real part is $1 - a^2 - b^2 > 0$ since $|w| < 1$. It is easy to see by the same reasoning that the points that go to the unit circle (i.e. with $a^2 + b^2 = 1$) are precisely those that lie on the imaginary axis (i.e. those z with $Re(z) = 0$). Note that the point -1 is covered by the "point at infinity," which by abuse of notation can be taken

on the imaginary axis (or any other line, which must "pass through" the point).
○—●

▶ $[\mathbf{10.21}]$: Show that any LFT is conformal on $\mathbb{C}$ if $c = 0$, or on $\mathbb{C} \setminus \{\frac{-d}{c}\}$ if $c \neq 0$.

●—○ The derivative can be computed by the quotient rule: if $T(z) = \frac{az+b}{cz+d}$, then

$$T'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

because $ad - bc \neq 0$. ○—●

▶ $[\mathbf{10.22}]$: Prove the above proposition.

●—○ Use the chain rule. ○—●

▶ $[\mathbf{10.23}]$: Show that the mapping f is conformal. Also show that it maps its domain onto the upper half-plane.

●—○ The derivative is computed by the chain rule, $\frac{d}{dz} z^{\pi/\alpha} = (\pi/\alpha) z^{\pi/\alpha - 1}$.
○—●

▶ $[\mathbf{10.28}]$: Show that T_{z_0} is conformal on U. Also, verify that T_{-z_0} is the inverse mapping for T_{z_0}, and that T_{z_0} is one-to-one.

●—○ LFTs are conformal on their domain and always one-to-one. The compositions $T_{-z_0} \circ T_{z_0}, T_{z_0} \circ T_{-z_0}$ can be computed directly; we leave it to the reader.
○—●

▶ $[\mathbf{10.30}]$: Let $D \subset \mathbb{C}$ be a simply connected open set. If $f : D \to \mathbb{C}$ is differentiable and $C \subset D$ is a closed contour, show that $\oint_C f(z) dz = 0$.

●—○ Immediate consequence of the general Cauchy integral theorem, Theorem 3.3 of Chapter 8. ○—●

▶ $[\mathbf{10.31}]$: Show that a star-shaped region is simply connected.

●—○ Let $C \subset D$ be a closed contour in the star-shaped region D. Now by definition, if $w \notin D$, then

$$n_C(w) = \frac{1}{2\pi i} \int_C \frac{dz}{z - w} = 0$$

by Cauchy's theorem for star-shaped reasons, because $\frac{1}{z-w}$ is analytic in D.
○—●

▶ $[\mathbf{10.32}]$: Use the maximum modulus theorem to finish the proof of the above lemma. Prove also that $|f'(0)| \leq 1$.

●—○ Indeed, $g(0) = f'(0)$, so since $|g(z)| \leq 1$ for all $z \in U$, the last claim is clear. ○—●

▶ $[\mathbf{10.34}]$: In the statement of the above theorem, why cant the set D be the whole complex plane? What would that imply?

•⊸ That would imply that there exists $f : \mathbb{C} \to U$ which is surjective, hence nonconstant; this contradicts Liouville's theorem. ⟜•