PEANO, RUSSELL, AND LOGICISM

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THE title is admittedly pretentious for a brief and perhaps unexciting point. Both are prompted by the fact that many philosophers and mathematicians are still concerned to argue about the achievement, or lack of such, of the Russellian programme for the " reduction " of mathematics to logic. Most such arguments, I believe, are based on a misconception about what Russell, and even Peano, accomplished. Cryptically put the confusion begins with the statement that Russell and Whitehead, after suitable definitions of Peano's primitive terms, deduced the Peano axioms.

Let ' P ' represent the Peano system (with five axioms and three undefined terms, ' zero ', ' number ', and ' successor ') ; ' PM ' stand for *Principia* ; ' C ' refer to a set of theorems of PM. Russell, after introducing three defined terms, deduced C in PM. But, though C may well be a particular interpretation of P, it cannot be identified with P. Russell's defined terms (' Zero ', ' Successor ', and ' Number ') are specific interpretations of the Peano primitives ; they do not supply definitions for Peano's primitive terms. In short, Russell did not deduce the Peano axioms in PM. Hence, one should not expect that since C can be obtained in PM all other interpretations of P can be obtained in PM. This point can be missed due to Russell's use of the same signs for both his defined terms and Peano's primitive terms.[1] But they are radically different, just as ' point ' taken as a primitive in an axiomatic system differs from its possible interpretations in terms of ' person ' or ' physical point '—to recall some elementary examples. The overlooking of this simple consideration can lead to controversy over the extent of Russell's success, with particular reference to the various domains of modern algebra, *e.g.*, it is argued that one cannot " get " groups, vector spaces, etc., Russellwise.

Upon suitable interpretations (in terms of the integers, zero, and plus, for example) of the elements and operations, specific interpretations of the group axioms can be derived in PM. Or, to put it another way, there are arithmetical images (or models) of the group axioms. This does not mean, nor should it, that

[1] B. Russell, *Introduction to Mathematical Philosophy*, pp. 9–28.

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all such interpretations can be so derived nor does it mean that the group axioms can be so derived. But this is true, as we saw, for the Peano axioms as well as for the group axioms. Further, this is no "*limitation*" *of PM* in any reasonable sense of that term. It reflects a generality for all such types of "reduction". Given two axiomatic systems, I and II, where II contains primitives not in I, upon the introduction of certain (defined) terms into I it may appear that the axioms of II can be deduced as theorems of I. Actually the propositions deduced in I constitute an interpretation of II and not the system itself. The same point may be made about the achievement of Peano in the construction of the systems of integers, rationals, real numbers, etc. If we consider all such systems as separate axiomatic systems —N_1, N_2, N_3, etc.—then from P one deduces not N_1 but an interpretation of N_1, and so on "up". Again, one should not be surprised if one does not get all such interpretations.

Consequently, to say that one cannot get the axioms for groups or vector spaces from PM is, at best, to say something true and trivial. For, to put the logistic thesis in the light of these remarks, one should never expect to get such axiomatic systems Russellwise. To put it even stronger—it makes no sense to say that we can so get them.

The situation may be clarified by a comparison. Consider a Euclidean plane geometry (E) as an axiomatic system and the system (E′) that results from an interpretation of E in terms of the real number system. The axioms and theorems of E′ (with a Russellian treatment of the interpreting terms) are logical truths. But we would not want to say that the axioms of E are tautologies; consequently, we would not say that E has been "reduced" either to logic or the real number system. The critical question is whether or not this latter situation is like the one between P and PM. In the respects relevant to this discussion I think it is. The similarity is bypassed by identifying the undefined terms 'zero', etc., of P with the defined terms 'Zero' etc., of PM. But, as stated above, there is no such identity; one set of terms simply provides an interpretation of the other set just as in the case of E and E′. The fact that on a particular interpretation we get propositions that are logical truths in no way implies that the original propositions are logical truths. The point may be reinforced by considering that just as we can have the interpretations of E that do not turn the axioms into tautologies, we can have, without too much stretch of the imagination, similar interpretations of P. Just as we have many instances of arithmetical interpretations of axiomatic systems which would never tempt us to state that such axiomatic systems have been reduced to arithmetic, we should not be tempted to state, in view of interpretations of some axiomatic systems in terms of logistic systems, that such axiomatic systems have been *reduced* to logic. The Russell-Whitehead accomplishment provides a specific interpretation of an axiomatic system in which the interpreted propositions are tautologies.

By itself this achievement does not provide an explication of the idea that arithmetical truths are tautologies. Consequently, one may ask whether or not this particular interpretation can serve as the basis for an adequate explication of our arithmetical notions. That is, can a formally constructed logistic system be employed as a tool for the analysis of our ordinary arithmetical concepts? The question thus boils down to the attempt to explicate terms of our ordinary language via constructions in certain types of formal languages. Hence, however we answer the question of the adequacy of Russell's achievement, the attempt to do so must consist of an analysis, in our ordinary language, of certain terms in formal languages and their supposed counterparts in ordinary language. The critical problems about the logistic thesis are thus neither mathematical nor formal.

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